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# OPERATOR DUALS OF SOME SEQUENCE SPACES

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The object of the present paper is to determine some generalized Köthe-Toeplitz duals which generalize and unify corresponding earlier results of Maddox.

§1. Throughout the paper we take $E$ and $F$ to be complex seminormed spaces and $(A_k)$ to be a sequence of linear operators on $E$ into $F$. Throughout $P = (p_k)$ denotes a strictly positive sequence of elements and $U$ denotes the unit ball in $E$. The following notations are used :

$B(E, F)$ = the space of continuous linear operators on $E$ into $F$.

$W(E)$ = {the space of all sequences of elements of $E$}

$$\underline{C_o}(p, E) = \{x = (x_k) \in W(E) \mid \|x_k\|^{p_k} \to 0 \quad \text{as } k \to \infty\}$$

$$C(p, E) = \{x = (x_k) \in W(E) \mid \text{there exists } l \in E \text{ with}$$

$$\|x_k - 1\|^{p_k} \to 0 \quad \text{as } k \to \infty\}$$

$$l(p, E) = \{x = (x_k) \in W(E) \mid \sum_k \|x_k\|^{p_k} < \infty\}.$$

If we take $E = F = C$, the set of complex numbers, these spaces reduce to the already familiar spaces $\underline{C_o}(p)$, $c(p)$ and $l(p)$ respectively. The most common of these spaces are $\underline{C_o}(E)$, $C(E)$ and $l(E)$ which we obtain by putting $E = F$ and $p_k = 1$ for all $k \geqslant 1$.

The $\alpha$ - and $\beta$-duals of Köthe were generalized by Robinson[6] who replaced scalar sequences by sequences of linear operators. Accordingly, we define $\alpha$ and $\beta$ duals of a subspace $X$ of $W(E)$ by

$$X^\alpha = \{(A_k) \mid \sum_k \|A_k x_k\| \text{ converges for all } x = (x_k) \in X\}$$

$$X^\beta = \{(A_k) \mid \sum_k A_k x_k \text{ converges in } F \text{ for all } x = (x_k) \in X\}.$$

Clearly $X^\alpha \subset X^\beta$ if $F$ is complete and the inclusion may be strict.

In Theorem 1, we compute $Co\,(p, E)$ for all $p_k = O\,(1)$, which includes the duals $Co^\alpha\,(E)$ and $Co^\alpha\,(p)$, computed earlier by Maddox[2,4]. Similarly generalising earlier results, the duals $l^\beta\,(p, E)$ for $p_k \leqslant 1$, $l^\alpha\,(p, E)$ for $p_k \geqslant 1$, $C^\alpha\,(p, E)$ and $l^\alpha_\infty\,(p, E)$ are obtained in Theorems 3, 4, 2 and 5 respectively.

§2. The following lemmas are used in providing the theorems of this paper.

*Lemma* 1—If $p > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then for any complex numbers $x$ and $y$

$$| xy | \leqslant | x |^q N^{-q/p} + N | y |^p$$

where $N$ is any positive integer.

This follo ws from the well known inequality, (see for instance Hardy and Littlewood[1], p. 17)

$$a^\alpha b^\beta \leqslant \alpha a + \beta b,\ 0 < \alpha,\ 1 \geqslant \beta,\ 0 \leqslant a,\ 0 \leqslant b,\ \alpha + \beta = 1$$

by putting

$$\alpha = 1/p,\ \beta = 1/q,\ a^\alpha = N^{1/p} | y | ,\ b^\beta = N^{-1/p} | x | .$$

*Lemma* 2, (Maddox[2], Theorem 1)—If $p_k$ 1, for all $k \geqslant 1$, then

$$l^\beta\,(p) = \bigcup_{N>1} \{x = (x_k) \mid \sum_k | x_k |^{q_k} N^{-q_k/p_k} < \infty\}$$

where $\dfrac{1}{p_k} + \dfrac{1}{q_k} = 1$ for all $k > 1$.

§3. This section contains the theorems dealt in the paper.

*Theorem* 1—Let $p_k = O\,(1)$. Then $(A_k) \in Co^\alpha\,(p, E)$ if and only if (i) there exists an integer $m > 1$ such that $A_k \in B\,(E, F)$ for each $k \geqslant m$ and (ii) there exists an integer $N > 1$ such that

$$\sum_{k \geqslant m} \| A_k \| N^{-1/p_k} < \infty.$$

Proof : Let $x = (x_k) \in Co\,(p, E)$ and (i), (ii) be true. Then there exists an integer $k_1$ such that

$$\| x_k \|^{p_k} < \frac{1}{N}$$

for all $k \geqslant k_1$ and there exists an integer $k_2 \geqslant m$ such that

$$\sum_{k \geqslant k_2} \| A_k \| N^{-1/p_k} < \epsilon$$

for a given $\epsilon > 0$. Put $K_0 = \max(k_1, k_2)$, so that

$$\sum_{k \geqslant k_0} \| A_k x_k \| \leqslant \sum_{k \geqslant k_0} \| A_k \| \|x_k\| \leqslant \sum_{k \geqslant k_0} \| A_k \| \; N^{-1/p_k} < \epsilon$$

and hence $(A_k) \in Co^\alpha(p, E)$.

Conversely, suppose that $(A_k) \in Co^\alpha(p, E)$. If (i) does not hold, we can choose integers $1 < k_1 < k_2 < \ldots\ldots$ such that $A_{k_i}$ is not bounded for each $i \geqslant 1$. Hence there exists $Z_{k_i} \in U$ such that

$$\| A_{k_i} \; Z_{k_i} \| > i^{1/p_{k_i}} \qquad \ldots(1)$$

for each $i \geqslant 1$. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} Z_{k_i} / i^{1/p_{k_i}}, \text{ if } k = k_i \quad \text{for each } i \\ 0, \text{ otherwise.} \end{cases}$$

Clearly, $x \in Co(p, E)$, but by (1)

$$\| A_{k_i} \; x_{k_i} \| > 1$$

for each $i \geqslant 1$, so that $\sum_k A_k x_k$ diverges, which leads to a contradiction.

To establish (ii) assume that

$$\sum_{k \geqslant m} \| A_k \| N^{-1/p_k} = \infty$$

for each integer $N > 1$. Hence we can choose integers $m = k_0 < k_1 < k_2 < \ldots\ldots$, such that

$$M_s = \sum_{k_{s-1} \leqslant k < k_s} \| A_k \| \, s^{-1/p_k} > 1$$

for each $s \geqslant 1$. By (i), for each $k \geqslant m$ there exists $Z_k \in U$ such that

$$\| A_k Z_k \| > \tfrac{1}{2} \| A_k \|.$$

Define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \dfrac{s^{-1/p_k}}{M_s} \quad Z_k, \; k_{s-1} \leqslant k < k_s' \quad \text{for each } s \geqslant 1 \\ 0, \quad \text{otherwise} \end{cases}$$

so that $x \in Co(p, E)$, since

$$\| x_k \|^{p_k} \leqslant \frac{s^{-1}}{M_s} < s^{-1} \to 0, \quad \text{as } k \to \infty.$$

However,

$$\sum_k \| A_k x_k \| = \sum_s \sum_{k_{s-1} \leqslant k < k_s} \frac{s}{M_s}^{-1/p_k} \| A_k Z_k \|$$

$$> \sum_s \frac{1}{M_s} \sum_{k_{s-1} \leqslant k < k_s} s^{-1/p_k} \frac{\| A_k \|}{2}$$

$$> \tfrac{1}{2} \sum_k 1 = \infty$$

which contradicts our assumption that $(A_k)$ $Co\,(p, E)$, and hence the theorem is established.

The following results are deducible from Theorem 1.

*Corollary* 1, (Maddox[4], Proposition 3.4)—The sequence $(A_k) \in Co^{\alpha}\,(E)$ if and only if there exists an integer $m \geqslant 1$ such that (i) $A_k \in B\,(E, F)$ for all $k \geqslant m$ and (ii) $\sum_{k \geqslant m} \| A_k \| < \infty$, where $E$, $F$ are Banach spaces.

Also we note that the argument of Theorem 6 (Maddox[2]) shows that, for $p_k > 0$, the $\beta$ -dual of $Co\,(p)$ is equal to its $\alpha$ -dual. Hence we have the following result.

*Corollary* 2—$Co^{\alpha}\,(p) = \bigcup_{N>1} \{a = (a_k) \mid \sum_k \mid a_k \mid N^{-1/p_k} < \infty\}$.

*Corollary* 3—If $p_k = O\,(1)$ and $\inf_k p_k > o$

then

$$Co\,(p) = l_1.$$

PROOF : Let $M = \sup_k p_k$ and $L = \inf_k p_k$.

We now proceed to show that under these two given conditions $Co\,(p) = Co$ and since, $Co^{\alpha} = l_1$, (Maddox[4], p. 19, 22), there is nothing to prove. Let $x = (x_k) \in Co\,(p)$. Then for every $0 < \epsilon < 1$, there exists an integer $k_0$ such that

$$\mid x_k \mid^{p_k} < \epsilon^M$$

for all $k \geqslant k_0$. Hence,

$$\mid x_k \mid^{p_k} < \epsilon^{M/p_k} < \epsilon$$

for all $k \geqslant k_0$, which shows that $x \in Co$. Conversely, if $x = (x_k) \in Co$, then for every $0 < \epsilon < 1$, there exists an integer $k_1$ such that

$$\mid x_k \mid < \epsilon^{1/p}$$

for all $k > k_1$. Hence,

$$| x_k |^{p_k} < \epsilon^{p_k/p} < \epsilon$$

for all $k \geq k_0$, which shows that $x \in Co(p)$.

Consequently

$$Co = Co(p).$$

This proves our assertion.

*Theorem* 2—Let $p_k = O(1)$. Then $(A_k) \in C^\alpha(p, E)$ if and only if (i) there exists an integer $m \geq 1$ such that $(A_k) \in B(E, F)$ for all $k \geqslant m$, (ii) there exists an integer $N > 1$ such that $\sum_{k \geqslant m} \| A_k \| N^{-1/p_k} < \infty$ and (iii) $\sum_k A_k Z$ converges absolutely in $F$ for each $Z \in E$.

PROOF : For the sufficiency let $x_k \to l[C(p, E)]$ as $k \to \infty$. Then $(x_k - 1) \in Co(p, E)$ and hence by Theorem 1

$$\sum_k \| A_k (x_k - 1) \| < \infty.$$

Since by (iii) $\sum_k \| A_k l \| < \infty$

$$\sum_k \| A_k x_k \| < \infty$$

which shows that

$$(A_k) \in C^\alpha(p, E).$$

Conversely, let $A(k) \in C^\alpha(p, E)$. Since $C^\alpha(p, E) \subset C^\alpha o(p, E)$, (i) and (ii) follow immediately from Theorem 1. To prove (iii) take any $Z \in E$. Then the sequence $(Z, Z, \ldots) \in C(p, E)$ and so $\sum_k \| A_k Z \| < \infty$.

This completes the proof of Theorem 2.

*Corollary* 4 (Maddox[4], p. 23)—$C^\alpha(E) = Co^\alpha(E) = l_\infty^\alpha(E)$.

*Corollary* 5—Let $p_k = O(1)$ and $\inf_k p_k > 0$. Then $C^\alpha(p) = l_1$.

PROOF : Let $(a_k) \in l_1$ and $x_k \to l[C(p)]$. Then $(x_k - 1) \in Co(p)$. Then by Corollary 3

$$\sum_k | a_k (x_k - 1) | < \infty$$

and since

$$(a_k) \in l_1, \quad \sum_k \mid a_k \, l \mid < \infty$$

which implies that

$$\sum_k \mid a_k \, x_k \mid < \infty.$$

Conversely, if $(a_k) \in C^\alpha(p)$, then consider the sequence $(1, 1, 1, \ldots) \in C(p)$. Then $\sum_k \mid a_k \mid < \infty$ follows immediately, so that $(a_k) \in l_1$. This establishes the corollary.

*Theorem* 3—Let $p_k \leqslant 1$ for each positive integer $k \geqslant 1$ and $F$ be a complete seminormed space. Then $(A_k) \in l^\beta(p, E)$ if and only if (i) there exists an integer $m$ such that $A_k \in B(E, F)$ for all $k \geqslant m$ and (ii) $\sup_{k \geqslant m} \| A_k \|^{p_k} < \infty$.

PROOF : Let $x = (x_k) \in l(p, E)$ and (i), (ii) be true.

Put

$$\delta = \sup_k \| A_k \|^{p_k}$$

and assume $\delta > 0$, otherwise, there is nothing to prove. Clearly there exists an integer $K_1$ such that

$$\sum_{k \geqslant k_1} \| x_k \|^{p_k} < \frac{1}{\delta} .$$

Then

$$\| A_k \, x_k \|^{p_k} \leqslant \| A_k \|^{p_k} \; \| x_k \|^{p_k} < 1$$

for all $k \geqslant k_1$, implying that $\| A_k \, x_k \| < 1$ for such $k$, and hence

$$\sum_{k \geqslant k_1} \| A_k \, x_k \| \leqslant \sum_{k \geqslant k_1} \| A_k \, x_k \|^{p_k} \leqslant \delta \sum_{k \geqslant k_1} \| x_k \|^{p_k}$$

so that $(A_k) \in l^\beta(p, E)$, since $F$ is complete.

Conversely, if (i) fails, we can choose an increasing sequence $(k_i)$ of integers such that $A_{k_i}$ is not bounded for each $i \geqslant 1$. So for such $i$, there exists $Z_{k_i} \in U$ such that

$$\| A_{k_i} \; Z_{k_i} \| > i^{2/p_{k_i}}.$$

Define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} Z_{k_i}/i^{2/p_{k_i}}, & \text{if } k = k_i, \text{ for each } i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_k \| x_k \|^{p_k} = \sum_i \| Z_{k_i}/i^{2/p_{k_i}} \|^{p_{k_i}} \leqslant \sum_i i^{-2} < \infty$$

so that

$x \in l(p, E)$, but $\| A_{k_i} x_{k_i} \| > 1$, for each $i$, implying that $\sum_k A_k x_k$ diverges in $F$, contrary to the assumption that $(A_k) \in l^\beta(p, E)$. To prove (ii) assume that $\sup_{k>m} \| A_k \|^{p_k} = \infty$, for every fixed $m = 1, 2, \ldots\ldots$ .

Choose integers $m = k_0 < k_1 < k_2 < \ldots\ldots\ldots$ with

$$\| A_{k_s} \|^{p_{k_s}} > s^2 \qquad \ldots(2)$$

for each $s \geqslant 1$. By (i) there exists $Z_{k_s} \in U$ such that

$$\| A_{k_s} \| \leqslant 2 \| A_{k_s} Z_{k_s} \| \qquad \ldots(3)$$

for each $s \geqslant 1$. We define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} Z_{k_s}/\| A_{k_s} \|, & \text{if } k = k_s \text{ for each } s > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $x \in l(p, E)$, but

$$\| A_{k_s} x_{k_s} \| > \tfrac{1}{2}$$

by (3), implying that $\sum_k A_k x_k$ diverges in $F$, which establishes (ii).

This completes the proof of Theorem 3.

If $E$ is a complete seminormed space then we have the following Corollaries :

*Corollary* 6—If $p_k \leqslant 1$, then $l^\alpha(p, E) = l^\beta(p, E)$.

*Corollary* 7 (Maddox[4], p. 27). If $0 < p \leqslant 1$, then

$$l_p^\alpha(E) = l_p^\beta(E).$$

*Corollary* 8, (Simons[7], Theorem 7)—If $p_k \leqslant 1$, then $l^{\beta}(p) = l_{\infty}(p)$.

Next we consider the $\alpha$-duals of $l(p, E)$, where $p_k > 1$ for all $k \geqslant 1$ and $l_{\infty}(p, E)$. The $\beta$-dual of these two spaces are still open for investigation.

*Theorem* 4—Let $p_k > 1$ for all $k \geqslant 1$, $p_k = O(1)$. Then $(A_k) \in l^{\alpha}(p, E)$ if and only if (i) there exists an integer $m \geqslant 1$ such that $A_k \in B(E, F)$ for all $k \geqslant m$ and (ii) there exists an integer $N > 1$ such that

$$\sum_{k \geqslant m} \| A_k \|^{p_k} N^{-p_k} < \infty$$

where

$$\frac{1}{p_k} + \frac{1}{q_k} = 1$$

for each $k \geqslant 1$.

Proof : Let (i), (ii) be true. Choose $n > s \geqslant m$. Then for any $x \in l(p, E)$.

$$\sum_{k=s}^{n} \| A_k x_k \| \leqslant \sum_{k=s}^{n} \| A_k \| \| x_k \|$$

$$\leqslant \sum_{k=s}^{n} \left[ N^{-q_k/p_k} \| A_k \|^{q_k} + N \| x_k \|^{p_k} \right]$$

$$= N \sum_{k=s}^{n} N^{-q_k} \| A_k \|^{q_k} + N \sum_{k=s}^{n} \| x_k \|^{p_k}$$

by Lemma 1, so that $\sum_k \| A_k x_k \| < \infty$, and hence $(A_k) \in l^{\alpha}(p, E)$.

Conversely, let $(A_k) \in l^{\alpha}(p, E)$. Since $l_1(E) \subset l(p, E)$ for $p_k \geqslant 1$, (i) holds by Corollary 6 of Theorem 3. To prove (ii) for each $k \geqslant m$ choose $Z_k \in U$ such that

$$\| A_k \| \leqslant 2 \| A_k Z_k \|.$$

Let $\lambda = (\lambda_k) \in l(p)$. Then $(\lambda_k Z_k) \in l(p, E)$ and hence

$$\sum_k \| A_k (\lambda_k Z_k) \| = \sum_k | \lambda_k | \| A_k Z_k \| < \infty$$

whenever $\lambda \in l(p)$, implying that $(\| A_k Z_k \|) \in l^{\alpha}(p)$.

Hence by Lemma 2, there exists an integer $N_1 > 1$ such that

$$\sum_k \| A_k Z_k \|^{q_k} N_1^{-q_k} < \infty.$$

Put $N = 2N_1$, so that

$$\sum_{k \geqslant m} \| A_k \|^{q_k} N^{-q_k} \leqslant \sum_{k \geqslant m} 2^{q_k} \| A_k Z_k \|^{q_k} 2^{-q_k} N_1^{-q_k}$$

$$= \sum_{k \geqslant m} \| A_k Z_k \|^{q_k} N_1^{-q_k} < \infty.$$

Which establishes (ii).

This completes the proof of Theorem 4.

*Corollary* 9—Let $1 < \inf_k p_k$, $p_k = O(1)$. Then $(A_k) \in l^\alpha(p, E)$ if and only if (i) there exists an integer $m \geqslant 1$ such that $A_k \in B(E, F)$ for all $k \geqslant m$ and (ii)

$$\sum_{k \geqslant m} \| A_k \|^{q_k} < \infty, \text{ where } \frac{1}{p_k} + \frac{1}{q_k} = \text{for all } k \geqslant 1.$$

*Corollory* 10, (Maddox[4], Proposition 3.9)—Let $1 < p < \infty$. Then $(A_k) \in l_p^\alpha(E)$ if and only if (i) there exists an integer $m \geqslant 1$ such that $A_k \in B(E, F)$ for all $k \geqslant m$ and (ii)

$$\sum_{k \geqslant m} \| A_k \|^{q} < \infty, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

*Theorem* 5—The sequence $(A_k) \in l_\infty^\alpha(p, E)$ if and only if (i) there exists an integer $m \geqslant 1$ such that $A_k \in B(E, F)$ for all $k \geqslant m$ and (ii)

$$\sum_{k \geqslant m} \| A_k \| N^{1/p_k} < \infty \text{ for each integer } N > 1.$$

PROOF : Let (i) and (ii) be true and $x \in l_\infty(p, E)$. If $N$ is an integer exceeding $\max(1, \sup_k \| x_k \|^{p_k})$, then

$$\sum_{k \geqslant m} \| A_k x_k \| \leqslant \sum_{k \geqslant m} \| A_k \| N^{1/p_k} < \infty$$

by (ii), so that $\sum_k \| A_k x_k \|$ converges, implying that $(A_k) \in l_\infty^\alpha(p, E)$.

Conversely, let $(A_k) \in l_\infty^\alpha(p, E)$. Since $Co(p, E) \subset l_\infty(p, E)$, (i) follows immediately from Theorem 1. To prove (ii) let

$$\sum_{k \geqslant m} \| A_k \| N^{1/p_k} = \infty \qquad \ldots(4)$$

for at least one integer $N > 1$. By (i), there exists $Z_k \in U$ such that

$$\| A_k \| \leqslant 2 \| A_k Z_k \|$$

for each $k \geqslant m$. Define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} N^{1/p_k} Z_k, & \text{if } k \geqslant m \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $x \in l_\infty(p, E)$, but

$$\sum_{k \geqslant m} \| A_k x_k \| = \sum_{k \geqslant m} N^{1/p_k} \| A_k Z_k \| > \tfrac{1}{2} \sum_{k \geqslant m} N^{1/p_k} \| A_k \| = \infty$$

by (4), which contradicts our assumption that $(A_k) \in l_\infty^\alpha(p, E)$. This establishes (ii) and hence the proof of Theorem 6 is complete.

*Corollary* 11 (Maddox[4], p. 23)—$l_\infty^\alpha(E) = Co^\alpha(E)$.

*Corollary* 12—$l_\infty^\beta(p) = \bigcap_{N>1} \{x = (x_k) \mid \sum_k | x_k | N^{1/p_k} < \infty\}$.

*Remarks* : It is also to possible to show that (Rath[5], Theorem 5) if $0 < p_k < 1$, then

$$w^\alpha(p, E) = \bigcup_{N>1} \{(A_k) \mid \sum_{r \geqslant m} \max_r | (2^r N^{-1})^{1/p_k} \| A_k \| | < \infty\}$$

where $m$ is an integer such that $A_k \in B(E, F)$ for all $k \geqslant m$ and

$$w(p, E) = \{x = (x_k) \in (E) \mid \text{there exists } l \in E \text{ with } 2^{-r} \Sigma_r \| x_k - l \|^{p_k} \to 0 \text{ as } r \to \infty.$$

where $\max_r$ and $\Sigma_r$ denote maximum and summation over $2^{r-1} \leqslant k < 2^r$ respectively. This also includes the duals $w^\alpha(p)$ and $w_p^\alpha(E)$, computed earlier by Maddox and Lascarides[3] and Maddox[4]. The case in which $p_k > 1$ is still open, except when $p_k = p > 1$ for all $k \geqslant 1$, see for instance[5].

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# ON STRONGLY NBD-FINITE FAMILIES

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The purpose of this paper is to give the properties of strongly nbd-finite families of Topological Spaces.

## Introduction

The concept of strongly nbd-finite family of a topological space has been introduced in Thangavelu[6] in order to study the piecewise definition of certain maps such as irresolute[1], semicontinuous[4] etc. The purpose of the present paper is to obtain several properties of strongly nbd-finite families. In fact Theorem 2.10 is revised as per the suggestions given by the referee. The concepts of semiopen set, semiclosed set, $\alpha$-set (Alpha set) and Semiclosure are discussed respectively in Levine[4], Crossley[1], Njastad[5] and Crossley[1].

## 1. Preliminaries

By $X$ we always mean a topological space on which no separation axioms are assumed unless explicitly stated. If $A$ and $B$ are subsets, $A - B$ denotes the difference of $B$ in $A$. Listed below are definitions and results that will be utilized in this paper.

*Definition* 1.1[6]—A family $\{A_m : m \in M\}$ of Subsets of $X$ is said to be strongly nbd-finite if for each $x$ in $X$, there is an open set $V$ containing $x$, satisfying one of the following conditions :

(a) $V \cap A_m = \phi$ for every $m \in M$

(b) There is a non-empty finite subset $N$ of $M$ such that

(i) $V \cap A_m \neq \phi$ for every $m \in N$

(ii) $V \cap A_m \subset A_k$ for every $m$, $k$ with $m \in N$, $K \in N$ and

(iii) $V \cap A_m = \phi$ for every $m \in M - N$.

*Theorem* 1.2[6]—If $\{A_m : m \in M\}$ is strongly nbd-finite then $\{\text{Scl}\,(A_m) : m \in M\}$ is also strongly nbd-finite in $X$ where Scl $(A)$ is the semiclosure of $A$, the largest semiclosed set that contains $A$.

*Theorem* 1.3[6]—The union of Semiclosed Sets of a strongly nbd-finite family is semiclosed.

*Definition* 1.4[2]—A space $X$ is semi-regular if and only if for each semiclosed set $A$ and $x \in X - A$, there exist disjoint Semiopen Sets $U$ and $V$ such that $x \in U$ and $A \subset V$.

*Theorem* 1.5[2]—$X$ is semi-regular if and only if for each $x$ and for each semi-open set $A$ containing $x$, there is a semi open set $B$ with $x \in B \subset \text{scl}(B) \subset A$.

## 2. Properties

From the Definition 1.1 it is obvious that a covering $\{A_m : m \in M\}$ of $X$ is strongly nbd-finite if and only if the condition (1.1) (b) holds. Further a subfamily of a strongly nbd-finite family is also strongly nbd-finite and if $\{A_m : m \in M\}$ is strongly nbd-finite in $X$ then $\{A \cap A_m : m \in M\}$ is also strongly nbd-finite in the subspace $A$ of $X$.

The proof of the next Theorem is analogous to that of Theorem 1.2.

*Theorem* 2.1—If $\{A_m : m \in M\}$ is strdngly nbd-finite then $\{\text{Cl}(A_m) : m \in M\}$ is also strongly nbd-finite where $\text{Cl}(A)$ is the closure of $A$.

*Theorem* 2.2—Let $\{A_m : m \in M\}$ be any family of sets in $X$. If the Union of semiclosures of $A_m$, $m \in M$ is semiclosed then it is equal to the semiclosure of the Union of $A_m$, $m \in M$.

Proof : Follows from the properties of Semiclosure.

*Corollary* 2.3—If $\{A_m : m \in M\}$ is any strongly nbd-finite family of sets then $\cup \{\text{scl}(A_m) : m \in M\} = \text{scl}(\cup \{A_m : m \in M\})$.

Proof : Follows from Theorems 1.2, 1.3 and 2.2.

*Theorem* 2.4—If $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are strongly nbd-finite families of $X$ then $\{A_i \cap B_j : (i, j) \in I \times J\}$ is also strongly nbd-finite.

Proof : Fix $x$ in $X$. There are open sets $V_1$ and $V_2$ both containing $x$ and satisfying the following conditions :

*Condition* 2.5—(a) $V_1 \cap A_i = \phi$ for every $i$ in $I$:

or

(b) There is a non-empty finite subset $N_1$ of $I$ such that (i) $V_1 \cap A_i \neq \phi$ for every $i$ in $N_1$, (ii) $V_1 \cap A_i \subset A_k$ for every $i, k$ with $i \in N_1$, $K \in N_1$ and (iii) $V_1 \cap A_i = \phi$ for every $i \in I - N_1$.

*Condition* 2.6—(a) $V_2 \cap B_j = \phi$ for every $j \in J$

or

(b) There is a non-empty finite subset $N_2$ of $J$ such that (i) $V_2 \cap B_j \neq \phi$ for every $j \in N_2$, (ii) $V_2 \cap B_j \subset B_l$ for every $j$, $l$ with $j \in N_2$, $l \in N_2$ and (iii) $V_2 \cap B_j = \phi$ for every $j \in J - N_2$.

Let $V = V_1 \cap V_2$. Then $V$ is open and contains $x$. Now $I \times J = L_1 \cup L_2 \cup L_3 \cup L_4$ where $L_1 = N_1 \times N_2$, $L_2 = N_1 \times (J - N_2)$, $L_3 = (I - N_1) \times N_2$ and $L_4 = (I - N_1) \times (J - N_2)$. It is easy to verify that $V \cap A_i \cap B_j = \phi$ for every ordered pair $(i, j)$ in $L_2 \cup L_3 \cup L_4$. If this is also true for every pair $(i, j)$ in $L_1$ then the Theorem is proved. Suppose not, let $N = \{(i, j) \in L_1 : V \cap A_i \cap B_j \neq \phi\}$. Clearly $N$ is finite and non-empty. Let $(i, j)$ and $(k, 1)$ be in $N$. Then using Condition 2.5(b) and Condition 2.6 (b) we have $V \cap A_i \cap B_j = (V_1 \cap B_i) \cap (V_2 \cap B_j) \subset A_k \cap B_1$. This proves the Theorem.

If a covering of $X$ has a nbd-finite refinement then it also has a precise nbd-finite refinement. The next Theorem is an analog of this.

*Theorem* 2.7—If the covering $\{A_i : i \in I\}$ of $X$ has a strongly nbd-finite refinement $\{B_m : m \in M\}$ then it also has a precise strongly nbd-finite refinement $\{C_k : k \in I\}$. Further more if each $B_m$ is open (resp. Semiopen, resp. $\alpha$-set, resp. closed and resp. Semiclosed) then $C_k$ can be chosen to be open (resp. semiopen. resp. $\alpha$-set, resp, closed and resp. semiclosed).

Proof : For each $m$ in $M$ fix an $i \in I$ such that $B_m \subset A_i$. Having done this, define a map $F : M \to I$ defined by $F(m) = i$ if $B_m \subset A_i$. For each $i$ in $I$ take $C_i = \cup \{B_m : F(m) = i\}$. Clearly $C_i \subset A_i$ for each $i$ in $I$ and $\{C_k : k \in I\}$ is a covering of $X$ and hence a precise refinement of $\{A_i : i \in I\}$.

*Claim* : $\{C_k : k \in I\}$ is strongly nbd-finite.

Fix $x$ in $X$. Since $\{B_m : m \in M\}$ is a strongly nbd-finite covering of $X$, Choose $V$ as in 1.1 Satisfying 1.1 (b) where $A$ is replaced by $B$. Take $L = F(N) = \{F(m) : m \in N\}$. Clearly $L$ is finite and non-empty. If $i \in L$ then $F(m) = i$ for some $m \in N$ so that, using 1.1 (b) (i), $V$ intersects $B_m$ which implies $V$ intersects $C_i$. Let $i \in I - L$. Now $N \subset F^{-1}(F(N)) = F^{-1}(L)$. Taking complement, we get $M - N \supset F^{-1}(I - L)$ so that $F^{-1}(i) \subset M - N$ which implies, by using 1.1 (b) (iii), $V$ does not intersect $B_m$ for every $m$ with $F(m) = i$. This proves that $V \cap C_i = \phi$. Let $k \in L$. Now $V \cap C_k = V \cap (\cup \{B_m : F_{no} : F(m) = k\}) = \cup \{V \cap B_m : F(m) = k\}$ which is contained in $B_n$ for every $n \in N$ (by 1.1 (b) (ii)). This implies that $V \cap C_k \subset C_{F(n)}$ for every $n \in N$. This completes the proof of the claim.

If each $B_m$ is open (resp. semiopen, resp. $\alpha$-set) then $C_k$ is open (resp. semiopen, resp. $\alpha$-set). If each $B_m$ is closed, since every strongly nbd-finite family is nbd-finite and since union of closed sets of a nbd-finite family is closed, $C_k$ is closed for each $k$ in $I$. If each $B_m$ is semiclosed then, using, Theorem 1.3, $C_k$ is semiclosed for each $k$ in $I$. This proves the Theorem.

The next Theorem is the generalization of Theorem 1.5 of Dugundji[3] (p. 162).

*Theorem* 2.8—Let $\{E_i : i \in I\}$ be any family of Sets in a space $X$ and let $\{B_m : m \in M\}$ be a strongly nbd-finite Closed (resp. Semiclosed) pairwise disjoint covering

of $X$. Assume that each $B_m$ intersects at most finitely sets $E_i$. Then there is a strongly nbd-finite family $\{C(E_i) : i \in I\}$ such that $E_i \subset C(E_i)$ and $C(E_i)$ is clopen (resp. semiclopen), for each $i$ in $I$.

PROOF : For each $i \in I$, define $C(E_i) = X - \cup \{B_m : B_m \cap E_i = \phi\}$. Since $\{B_m : m \in M\}$ is pairwise disjoint $C(E_i) = \cup \{B_m : B_m \cap E_i \neq \phi\}$. If each $B_m$ is closed, since every strongly nbd-finite family is nbd-finite, each $C(E_i)$ is both open and closed. If each $B_m$ is semiclosed, by using Theorem 1.3 and Theorem 2.2, each $C(E_i)$ is both semiopen and semiclosed. Further, for each $i$ in $I$, it is easy to see that $C(E_i)$ intersects $B_m$ if and only if $E_i$ intersects $B_m$. Also $E_i \subset C(E_i)$ for each $i$ in $I$.

*Claim*—$\{C(E_i) : i \in I\}$ is strongly nbd-finite.

Fix $x$ in $X$. Since $\{B_m : m \in M\}$ is a strongly nbd-finite covering of $X$. Choose $V$ and $N$ as in 1.1 satisfying Definition 1.1 (b) where where $A$ is replaced by $B$. Since $\{B_m : m \in M\}$ is pairwise disjoint, $N$ is singleton namely $\{m_0\}$. Now using Definition 1.1 (b), $V \cap B_{m_0} \neq \phi$ and $V \cap B_m = \phi$ for all $m$ different from $m_0$. Since $\{B_m : m \in M\}$ is a covering, $V \subset B_{m_0}$. Since each $B_m$ intersects at most finitely many $E_i$, Choose a finite subset $N_0$ of $I$ such that $B_{m_0} \cap E_i \neq \phi$ for every $i \in N_0$ and $B_{m_0} \cap E_i = \phi$ for every $i \in I - N_0$.

If $i \in I - N_0$ then $B_{m_0} \cap E_i = \phi$ which implies $C(E_i) \cap B_{m_0} = \phi$ so that $V \cap C(E_i) = \phi$. Let $L$ denote the set of all $i$ in $N_0$ such that $V \cap C(E_i) \neq \phi$. If $L$ is empty then $V$ does not intersect $C(E_i)$ for each $i$ in $I$. Let $L$ be non-empty. As $N_0$ is finite, $L$ is finite. If $i, j$ are in $L$ then $V \cap C(E_i) = V \cap (\cup \{B_m : B_m \cap E_i \neq \phi\})$ which equals $V \cap B_{m_0} = V \subset B_{m_0}$.

Since $B_{m_0} \cap C(E_0) \neq \phi$, $B_{m_0} \subset C(E_j)$ so that $V \cap C(E_i) \subset C(E_j)$. This completes the proof.

*Theorem* 2.9—If $X$ is semi-regular (resp. regular) and if each semiopen (resp. open) covering has a strongly nbd-finite refinement then each semiopen (resp. open) covering has a semiclosed (resp. closed) strongly nbd-finite refinement.

PROOF : We prove for semi-regular $X$ and the proof for regular $X$ is similar.

Let $\{A_i : i \in I\}$ be a semiopen covering of $X$. By Theorem 1.5, for each $x$ in $X$ and for each $A_i$ with $x \in A_i$, there is a semiopen set $B_i$ with $x \in B_i \subset \text{Scl}(B_i) \subset A_i$. Now $\{B_i : i \in I\}$ is a semiopen covering for $X$. Therefore it has a strongly nbd-finite refinement $\{C_k : k \in K\}$. Then $\{\text{Scl}(C_k) : k \in K\}$ is a semiclosed strongly nbd-finite refinment of $\{A_i : i \in I\}$. This completes the proof.

*Theorem* 2.10—If each open covering of $X$ has an open refinement that can be decomposed into an at most countable collection of strongly nbd-finite families of open sets, then each open covering has a strongly nbd-finite refinement.

PROOF : Let $\{U_i : i \in I\}$ be an open covering of $X$ Then it has an open refinement $\{V_{n,m} : (n, m) \in Z^+ XM\}$ where $Z^+$ is the set of positive integers and for each fixed $n_0 \in Z^+$, the family $\{V_{n_0,m} : m \in M\}$ is strongly nbd-finite. For each positive integer $n \in Z^+$ let $W_n$ denote the Union of sets $V_{n,m}$ where $m$ varies over the index set $M$. Then $\{W_n : n \in Z^+\}$ is an open covering for $X$. Take $A_1 = W_1$ and $A_n = W_n - \bigcup_{K<n} W_k$ for $n \geqslant 2$. If $y \in X$, let $n(y)$ be the first $k$ such that $Y \in W_k$ so that $Y \in A_{n(y)}$. Therefore $\{A_n : n \in Z^+\}$ is a refinement of $\{W_k : k \in Z^+\}$. Now for each $x$ in $X$, $W_{n(x)} \cap A_{n(x)} \neq \phi$ and $W_{n(x)} \cap A_k = \phi$ if $k \neq n(x)$. Thus $\{A_n : n \in Z^+\}$ is strongly nbd-finite.

*Claim*—$\{A_n \cap V_{n,m} : (n, m) \in Z^+ X M\}$ is strongly nbd-finite.

Fix $x$ in $X$. Then $W_{n(x)} \cap A_k = \phi$ for every $k \neq n(x)$. Now $\{V_{n(x),m} - m \in M\}$ is strongly nbd-finite. Therefore there is an open set $G$ containing $x$ such that either $G \cap V_{n(x),m} = \phi$ for every $m$ in $M$ or there is a non-empty finite subset $\Delta$ of $M$ such that (i) $G \cap V_{n(x),m} \neq \phi$ for every $m$ in $\Delta$, (ii) $G \cap V_{n(x),m} \subset V_{n(x),k}$ for every $m, k$ with $m \in \Delta, k \in \Delta$ and (iii) $G \cap V_{n(x),m} = \phi$ for every $m$ in $M - \Delta$. Take $W = W_{n(x)} \cap G$. Let

$$N = \{(n(x), m) : W \cap A_{n(x)} \cap V_{n(x),m} \neq \phi,\ m \in \Delta\}.$$

If $N = \phi$ the claim is proved. So, let $N \neq \phi$. Clearly $N$ is finite. Let $(n(x), m)$ and $(n(x), k)$ be in $N$. Then $W \cap A_{n(x)} \cap V_{n(x),m} \subset G \cap A_{n(x)} \cap V_{n(x),m}$

$$\subset A_{n(x)} \cap V_{n(x),k}.$$

This completes the proof of the claim and hence the Theorem.

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# BOUNDED AND FRÉCHET DIFFERENTIALS FOR MAPPINGS ON LINEAR TOPOLOGICAL SPACES USING PSEUDONORM TOPOLOGY

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Conditions for the existence of bounded and Fréchet differentials for mappings on linear topological spaces are introduced using the pseudonorm topologies (see Hyers[3]) of the corresponding spaces. These results generalise the results of Václav Zizler[10] for mappings on normed linear spaces.

## 1. INTRODUCTION

The technique of pseudonorm topology, which can always be defined on every linear topological space (see Hyers[3], Theorem 9, p. 10), is used to introduce certain conditions (see Theorems 3.1 and 3.3) for the existence of bounded and Fréchet differentials for mappings on linear topological spaces to locally convex spaces. The concept of Fréchet differentiability for functions on normed linear spaces was extended by Hyers[3] for functions on linear topological spaces using pseudonorm topology. Initially Suchomlinov[7] introduced the concept of bounded differentials for functions on normed linear spaces to avoid assumption of linearity of the Fréchet differential. The authors extend here this notion (see Definition 2.3) for functions on linear topological spaces using pseudonorm topology. The proof uses the Lasalle pseudonorm topology[4] (see also, Hyers[3]) on the space of continuous linear operators along with Wehausen's[8] analogue to the particular case of Hahn Banach Theorem (see Theorem 2.1). Theorems 3.1 and 3.3 give in particular the results of Zizler[10] for a function on normed linear spaces. Earlier one, of the authors (Dayal and Jain[1], cor. 3.5, p. 481) introduced some conditions for existence of the Fréchet differentials generalising and improving the results of Nashed[6] and Marinescu[5].

## 2. PRELIMINARIES

Let $E$ be a linear topological (l. t.) space. $E$ can be regarded as a pseudonormed space (p. s.) (see Def. in Hyers[3] on p. 9 and Theorem 9, p. 10) with respect to certain directed system (d. s.) $D$. Further, if $E$ is locally convex, then the corresponding pseudonorm is triangular. $A$ subset $B$ of a p. s. $E$ with associated d. s. $D$ is said to be *bounded* if and only if given $d \in D$ there is a real number $\mu_d > 0$ (depending

on $d$) such that $| x |_d < \mu_d$ for all $x \in B$. The following theorem is due to Wehausen[8] and is an analogue of a corollary of Hahn Banach theorem for locally convex spaces.

*Theorem* 2.1—If $E$ is locally convex l. t. space with associated d. s. $D$, then for any $x_0 \in E$ and any $d \in D$ there exists a linear continuous functional $f$ defined on $E$ with the property that $| f(x_0) | = | x_0 |_d$.

From now onwards, we will mean by $E_1$ and $E_2$ l. t. spaces with associated directed systems $D_1$ and $D_2$ respectively. If we denote by $\mathscr{L}(E_1, E_2)$ the space of continuous linear operators from $E_1$ to $E_2$, one can introduce a pseudonorm on $\mathscr{L}(E_1, E_2)$ associated with the d. s. $\{(d_2, B) : d_2 \in D_2, 0 \in B \subseteq E_1, B \text{ bounded}\}$ (with usual ordering : $(d_2, B) \leqslant (d_2', B')$ if and only if $d_2 \leqslant d_2'$ and $B \subseteq B'$) by associating each $(d_2, B)$ to real number called pseudonorm of $T$, $| T |_{(d_2, B)}$, for every $T \in \mathscr{L}(E_1, E_2)$. This pseudonorm defines a topology on $\mathscr{L}(E_1, E_2)$ and is called Lasalle topology (see Lasalle[4] and also see Hyers[3]). The following theorem of Dayal (see Dayal and Jain[2], Theorem 2.4, p. 1098 for $n = 1$), using Lassalle topology, will be used later.

*Theorem* 2.2—Let $T : E_1 \to E_2$. If $T$ is continuous, then for every $d_2 \in D_2$, $B$ bounded subset of $E_1$ containing 0 and for every $d_1 \in D_1$ for which $| x |_{d_1} \neq 0$ if $x \neq 0$ there is a $\mu_{d_1} > 0$ such that

$$| T(x) |_{d_2} \leqslant \frac{| T |_{(d_2, B)}}{\mu_{d_1}} | x |_{d_1}.$$

*Definition* 2.3—Let $f : E_1 \to E_2$. $f$ is said to have Gâteaux differential $Vf(x_0, h)$ at $x_0 \in E_1$ if

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = Vf(x_0, h) \qquad \text{...(2.1)}$$

exists for every $h \in E_1$, in the sense that for every $d_2 \in D_2$ there is a $d_1 \in D_1$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on $d_2$, $d_1$, $\epsilon$ and $h$) with the property that

$$\left| \frac{f(x_0 + th) - f(x_0)}{t} - Vf(x_0, h) \right|_{d_2} < \epsilon \qquad \text{...(2.2)}$$

whenever $0 < | th |_{d_1} < \delta$.

If the limit (2.1) is uniform with respect to $h \in E_1$ in the sense that every $d_2 \in D_2$, there is a $d_1 \in D_1$ with the property that for every $\epsilon > 0$ there exists a $\delta > 0$ (depending on $d_2$, $d_1$ and $\epsilon$) such that (2.2) holds whenever $0 < | t | < \delta$; $| h |_{d_1} = 1$ and $Vf(x_0, h)$ is linear and continuous in $h$, then $f$ is said to have a 'Fréchet differential' $df\, x_0$ such $df(x_0, h) = Vf(x_0, h)$ for every $h \in E_1$.

If the limit (2.1) is uniform with respect to $h \in E_1$ and corresponding to every $d_2 \in D_2$ and $d_1 \in D_1$ there is $\mu_{d_1} > 0$ with the property that $| Vf(x_0, h) |_{d_2} < \mu_{d_1}$ for all $| h |_{d_1} = 1$, then $f$ is said to have 'bounded differential' $dVf(x_0, \ )$ at $x_0$ such that $dVf(x_0, h) = Vf(x_0, h)$ for every $h \in E_1$.

The formulation of the Fréchet differential in terms of the pseudonorm topology is given by Hyers[3] (p. 15) and the concept of bounded differential is an extension of the concept of bounded differential given by Suchomlinov[7], initially, for functions on normed linear spaces to avoid the linearity property.

## 3. Bounded Differentiability Conditions

We prove the following theorem for the existence of bounded differentials for mappings on linear topological spaces.

*Theorem* 3.1—Let $E_1$ and $E_2$ be linear topological spaces with associated directed systems $D_1$ and $D_2$ with the topology of $E_2$ being locally convex. Let $f : E_1 \rightarrow E_2$ have a Gâteaux differential $Vf(x, h)$ at every point $x$ in some neighbourhood of $x_0 \in E_1$ with the following conditions :

(i) Given $d_2 \in D_2$ there is a $d_1 \in D_1$ such that for every $\epsilon > 0$ there is a $\delta > 0$ (depending on $d_2, d_1, \epsilon$) with the property

$$| Vf(x_0 + th, h) - Vf(x_0, h) |_{d_1} < \epsilon \qquad \text{...(3.1)}$$

whenever $0 < | th |_{d_1} < \delta$ uniformly with respect to $h \in E_1, | h |_{d_1} = 1$.

(ii) Given $d_2 \in D_2$ there is a $d_1 \in D_1$ and a real number $\mu_{d_1} > 0$ such that

$$| Vf(x_0, h) |_{d_2} < \mu_{d_1} \text{ for all } h \in E_1, | h |_{d_1} = 1.$$

Then, $f$ possesses a bounded differential $dVf(x_0, h)$ at $x_0$ such that

$$dVf(x_0, h) = Vf(x_0, h).$$

The following mean value theorem, which can easily be proved by usual arguments (see Yamamuro[9], p. 15) will be used in the proof of main Theorem 3.1.

*Theorem* 3.2—Let $E$ be a linear topological space, $F$ be a locally convex space and $f : E \rightarrow F$ have a Gâteaux differential at every point of the line segment joining $x$ and $x + th$ in $E$. Then for every continuous linear functional $e$ on $F$ there exists a $\tau \in (0, 1)$ such that

$$< (f(x + th) - f(x)), e > = < tVf(x + \tau th, h), e >$$

where the notation $< y, e >$ means the value of $e$ on $y$ for every $y \in F$ and $e \in F^*$ the conjugate of $F$ i.e. $\{(F, R)$ with respect to Lassalt topology.

*Proof of Theorem* 3.1—Let $h$ be any arbitrary element in $E_1$ and write

$$w(x_0, th) = f(x_0 + th) - f(x_0) - Vf(x_0, th).$$

Since $f$ has a Gâteaux differential $Vf(x, h)$ at every point $x$ of some neighbourhood of $x_0$, given $d_2 \in D_2$ there exists $d_1 \in D_1$ such that for every $\epsilon > 0 \ \exists \ \delta > 0$ (depending on $d_2, d_1, \epsilon, h$) with the property that

$$\left| \frac{w(x_0, th)}{t} \right|_{d_1} < \epsilon$$

whenever $0 < | th |_{d_1} < \delta$ and $h \in E_1$. Suppose, for some $\epsilon > 0$, the choice of $\delta$ is not uniform with respect to $h \in E_1$, $| h |_{d_1} = 1$, then we may have a $d_2 \in D_2$ such that for every choice of $d_1 \in D_1$ and $n \in N$ there exists and $\epsilon > 0$, $h_n \in E_1$ with $| h_n |_{d_1} = 1$ with the property

$$\left| \frac{w(x_0, t_n h_n)}{t_n} \right|_{d_2} \geqslant \epsilon \qquad \ldots(3.2)$$

whenever $0 < | t_n h_n |_{d_1} < \dfrac{1}{n}$.

Let $\mathscr{L}(E_2, R)$ be the space of continuous linear functionals with the Lasalle[4] pseudonorm topology associated with the directed system $\{(|\ |, B) : B$ bounded in $E_1$ containing $0\}$. For $e_n \neq 0$ in $\mathscr{L}(E_2, R)$ we can have a bounded subset $B_n$ containing 0 in $E_2$ such that the pseudonorm $| e_n |_{(||, B_n)} \neq 0$ (cf. definition of pseudonorm, see Dayal and Jain[2], p. 1094) and a real number $\mu_{d_2} > 0$, associated with $B_n$ (cf. definition of bounded sets in §1.) By Gâteaux differentiability of $f$ at $x_0$, for $h \in E_1$ and $d_2 \in D_2$ there is a $d_1 \in D_1$ such that for the above chosen $\epsilon > 0$ there is an $n_0 \in N$ with the property that

$$\left| \frac{w(x_0, t_n h)}{t_n} \right|_{d_2} < \frac{\epsilon \cdot \mu_{d_2}}{3 \, | e_n |_{(||, B_n)}} \qquad \ldots(3.3)$$

whenever $n > n_0$.

Now

$$f(x_0 + t_n h) - f(x_0) = Vf(x_0, t_n h) + w(x_0, t_n h)$$

$$f(x_0 + t_n h_n) - f(x_0) = Vf(x_0, t_n h_n) + w(x_0, t_n h_n)$$

$$\begin{aligned} w(x_0, t_n h_n) = {} & w(x_0, t_n h) + (f(x_0 + t_n h_n) - f(x_0)) \\ & - (f(x_0 + t_n h) - f(x_0)) + (Vf(x_0, t_n h) \\ & - Vf(x_0, t_n h_n)). \end{aligned}$$

So we have

$$< w(x_0, t_n h_n), e_n > = < w(x_0, t_n h), e_n >$$
$$+ < (f(x_0 + t_n h_n) - f(x_0)), e_n >$$
$$- < (f(x_0 + t_n h) - f(x_0)), e_n >$$
$$+ < (Vf(x_0, t_n h) - Vf(x_0, t_n h_n)), e_n >.$$

By using the mean value theorem (Theorem 3.2)

$$< \frac{w(x_0, t_n h_n)}{t_n}, e_n > = < \frac{w(x_0, t_n h)}{t_n}, e_n >$$
$$+ < Vf(x_0 + \tau_n t_n h_n, h_n), e_n >$$
$$- < Vf(x_0 + \tau'_n t_n h, h), e_n >$$
$$+ < (Vf(x_0, h) - Vf(x_0, h_n)), e_n >$$

where $\tau_n$ and $\tau'_n$ are in $(0, 1)$.

Thus,

$$< \frac{w(x_0, t_n h_n)}{t_n}, e_n > = < \frac{w(x_0, t_n h)}{t_n}, e_n >$$
$$+ < (Vf(x_0 + \tau_n t_n h_n, h_n)$$
$$- Vf(x_0, h_n)), e_n >$$
$$+ < (Vf(x_0, h) - Vf(x_0 + \tau'_n t_n h, h)), e_n >.$$

By Theorem 3.2, for every bounded subset $B_n$ of $E_2$ containing 0 corresponding to each $n$ and every $d_2 \in D_2$ for which

$$\left| \frac{w(x_0, t_n h)}{t_n} \right|_{d_2} \neq 0 \text{ if } \frac{w(x_0, t_n h)}{t_n} \neq 0$$

$$| Vf(x_0 + \tau_n t_n h_n, h_n) - Vf(x_0, h_n |_{d_2} \neq 0$$

if

$$Vf(x_0 + \tau_n t_n h_n, h_n) - Vf(x_0, h_n) \neq 0$$

and

$$| Vf(x_0, h) - Vf(x_0 + \tau'_n t_n h, h) |_{d_2} \neq 0$$

if

$$Vf(x_0, h) - Vf(x_0 + \tau'_n t_n h, h) \neq 0$$

there is a real number $\mu d_2 > 0$ such that

$$\begin{aligned} \left| < \frac{w(x_0, t_n h_n)}{t_n}, e_n > \right| \leqslant & \frac{|e_n|_{(\|, B_n)}}{\mu d_2} \left| \frac{w(x_0, t_n h)}{t_n} \right|_{d_2} \\ & + \frac{|e_n|_{(\|, B_n)}}{\mu d_2} \mid Vf(x_0 + \tau_n t_n h_n, h_n) \\ & - Vf(x_0, h_n) \mid_{d_2} \\ & + \frac{|e_n|_{(\|, B_n)}}{\mu d_2} \mid Vf(x_0, h) \\ & - Vf(x_0 + \tau'_n h, h) \mid_{d_2} . \end{aligned}$$

In view of a theorem (2.1) Hyers[3] (p. 12) there exists an $e_n \in \mathscr{L}(E_2, R)$ such that

$$\left| < \frac{w(x_n, t_n h_n)}{t_n}, e_n > \right| = \left| \frac{w(x_0, t_n h_n)}{t_n} \right|_{d_2} .$$

Thus

$$\begin{aligned} \left| \frac{w(x_0, t_n h_n)}{t_n} \right|_{d_2} \leqslant \frac{|e_n|_{(\|, B_n)}}{\mu d_2} & \left\{ \left| \frac{w(x_0, t_n h)}{t_n} \right|_{d_2} \right. \\ & + | Vf(x_0 + \tau_n t_n h_n, h_n) - Vf(x_0, h_n) |_{d_2} \\ & \left. + | Vf(x_0, h) - Vf(x_0 + \tau'_n t_n h, h) |_{d_2} \right\} . \end{aligned} \quad \ldots(3.4)$$

In view of hypothesis (i) and expression (3.1), for $d_2 \in D_2$ there is a $d'_1 \in D_1$, which can be taken as $d_1$ in view of the def. of d. s. associated with a pseudonorm, such that for every $\epsilon > 0$ there exists $n_0 \in N$ such that for $n \geqslant n_0$, $n \in N$,

$$\begin{aligned} & | Vf(x_0 + \tau_n t_n h_n, h_n) - Vf(x_0, h_n) |_{d_2} \\ & \qquad + | Vf(x_0 + \tau'_n t_n h, h) - Vf(x_0, h) |_{d_2} \leqslant \frac{2\epsilon}{4} \cdot \frac{\mu d_2}{|e_n|_{(\|, B_n)}} . \end{aligned} \quad \ldots(3.5)$$

Hence (3.4) together with (3.5) and (3.3) contradicts (3.2). Thus

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0,)}{t} = Vf(x_0, h)$$

where the limit is taken uniformly with respect to $h \in E_1$.

The second condition for $Vf(x_0, h)$ being a bounded differential follows trivially by hypothesis (ii) of the theorem. This proves Theorem 3.1.

Trivially, since every normed linear space is a locally convex space and every norm is also a pseudonorm, Theorem 3.1 gives in particular the following theorem of Zizler[10].

*Theorem* [VZ]—Let $E_1$ and $E_2$ be normed linear spaces and $f : E_1 \to E_2$ has the Gâteaux differential $Vf(x, h)$ in some neighbourhood of $x_0$ in $E_1$. Let the following conditions be fulfilled

(i) $$\lim_{t \to 0} \| Vf(x_0 + th, h) - Vf(x_0, h) \|_{E_2} = 0$$

uniformly with respect to $\| h \| = 1, \quad h \in E_1$.

(ii) $Vf(x_0, h)$ is bounded on $S_1 = \{h : \| h \| = 1\}$.

Then $f$ possesses a bounded differential, in fact it is $Vf(x_0, h)$ at $x_0 \in E_1$.

*Remark* : It is easy to deduce that if $f: E_1 \to E_2$ is continuous in some neighbourhood of $x_0$ and the Gâteax differential $Vf(x_0, h)$ is linear in $h$ then $Vf(x_0, h)$ is continuous in $h$. Thus we have the following theorem :

*Theorem* 3.3—Let $E_1$ and $E_2$ be linear topological spaces with associated directed systems $D_1$ and $D_2$. Let $f : E_1 \to E_2$ be continuous in some neighbourhood of $x_0$. Suppose that the Gâteaux differential $Vf(x, h)$ exists in a neighbourhood of $x_0$ and $Vf(x_0, h)$ is linear in $h$ with the following condition that given $d_2 \in D_2$ there is a $d_1 \in D_1$ such that for every $\epsilon > 0$ there is a $\delta > 0$ (depending on $d_2, d_1, \epsilon$) with the property :

$$| Vf(x_0 + th, h) - Vf(x_0, h) |_{d_2} < \epsilon$$

whenever $0 < | th |_{d_1} < \delta$ uniformly with respect to $h \in E_1, | h |_{d_1} = 1$. Then the mapping $f$ has a Fréchet differential $df(x_0)$ at $x_0$ such that $df(x_0)(h) = Vf(x_0, h)$ for every $h \in E_1$.

Since every normed linear space is a locally convex space and every norm is also a pseudonorm, Theorem 3.3 gives in particular the following theorem of Zizler[10].

*Theorem* (VZ)—Let $f: E \to F$, $E$ and $F$ being normed linear space. $f : E \to F$

is continuous in some neighbourhood of $x_0$. Suppose that the Gâteaux differential $Vf(x, h)$ exists in a neighbourhood of $x_0$ and is such that $Vf(x_0, h)$ is linear in $h$ and

$$\lim_{t \to 0} \| Vf(x_0 + th, h) - Vf(x_0, h) \| = 0$$

holds uniformly with respect to $h \in E$, $\| h \| = 1$. Then $f$ has a Fréchet differential $df(x_0)$ at $x_0 \in E$.

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# LIE THEORY OF $q$-APPELL FUNCTIONS

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In this paper a generalized $q$-Appell function is defined and symmetry techniques for classification and derivation of generating functions are evolved. This is in analogy with the present Lie theoretic methods for ordinary Appell functions. We develop canonical system of $q$-difference equations for $q$-Appell functions and derive series identities using symmetry operators of these canonical equations.

## 1. INTRODUCTION

A Lie theoretic method was developed in Kalnins *et al.*[3] and Miller[4,5] which associated with each family of multivariable hypergeometric functions a canonical system of partial differential equations obtained from the differential recurrence relations obeyed by the family. The underlying idea was originally due to Weisner[7]. The hypergeometric functions arise by partial separation of variables in the cnnonical systems. Also any analytic solution of these equations can be regarded as a generating function for this family. Furthermore the symmetry operators for the canonical systems can be used to characterize these generating functions.

Later, an analogous theory for families of many variable basic hypergeometric functions was developed in Agarwal[1]. In this paper, we are going to single out $q$-Appell functions, define them in a unified form and study them further from Lie-theoretic view point. In the process we develop generating functions for the $q$-Appell polynomials too.

The symmetry techniques of this paper apply to former power series and are independent of convergence criteria. Hence the domains in which the derived identities are valid are not specified. Domains of validity for individual cases can be determined by the usual methods.

## 2. GENERALIZED $q$-APPELL FUNCTION

The $q$-Appell functions are defined as[2,6]

$$\text{(a)}\quad f_1\left(\begin{matrix} a, b, b' ; \\ c\ ; \end{matrix}\ x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}\,(b;q)_m\,(b';q)_n\,x^m\,y^n}{(c;q)_{m+n}\,(q;q)_m\,(q;q)_n}$$

$$\text{(b)}\quad f_2\left(\begin{matrix} a, b, b'; \\ c, c'; \end{matrix}\; x, y\right) = \sum_{m\cdot n=0}^{\infty} \frac{(a;q)_{m+n}\,(b;q)_m\,(b';q)_n\, x^m y^n}{(c;q)_m\,(c';q)_n\,(q;q)_m\,(q;q)_n}$$

$$\text{(c)}\quad f_3\left(\begin{matrix} a, a'\, b, b'; \\ c; \end{matrix}\; x, y\right) = \sum_{m\cdot n=0}^{\infty} \frac{(a;q)_m\,(a';q)_n\,(b;q)_m\,(b';q)_n\, x^m y^n}{(c;q)_{m+n}\,(q;q)_m\,(q;q)_n}$$

$$\text{(d)}\quad f_4\left(\begin{matrix} a, b\,; \\ c, c'; \end{matrix}\; x, y\right) = \sum_{m\cdot n=0}^{\infty} \frac{(a;q)_{m+n}\,(b;q)_{m+n}\, x^m y^n}{(c;q)_m\,(c';q)_n\,(q;q)_m\,(q;q)_n} \qquad \dots (2.1)$$

where $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$

and $a = q^{\alpha}$, $a' = q^{\alpha'}$, $b = q^{\beta}$, $b' = q^{\beta'}$, $c = q^{\gamma}$ and $c' = q^{\gamma'}$.

Instead of treating them separately, we define them in a unified form as

$$\mathscr{F}\left(\begin{matrix} a, a', b, b'; \\ c, c'; \end{matrix}\; x, y\right) = \sum_{m\cdot n=0}^{\infty} \frac{(a;q)_{m+dn}\,(a';q)_{d'n}\,(b;q)_{m+en}\,(b';q)_{e'n}\, x^m y^n}{(c;q)_{m+fn}\,(c';q)_{f'n}\,(q;q)_m\,(q;q)_n}. \qquad \dots (2.2)$$

Indeed

$$(d, d', e, e', f, f') = (1, 0, 0, 1, 1, 0) \Rightarrow \mathscr{F} = f_1$$

$$(d, d', e, e', f, f') = (1, 0, 0, 1, 0, 1) \Rightarrow \mathscr{F} = f_2$$

$$(d, d', e, e', f, f') = (0, 1, 0, 1, 1, 0) \Rightarrow \mathscr{F} = f_3$$

and

$$(d, d', e, e', f, f') = (1, 0, 1, 0, 0, 1) \Rightarrow \mathscr{F} = f_4. \qquad \dots (2.3)$$

Appropriately we name (2.2) as a generalized $q$-Appell function.

## 3. Canonical Forms

Let $T_u$ be the $q$-dilation operator corresponding to the variable $u$, that is, $T_u$ maps a function of the variables $u, v, w, \dots$ to the function

$$T_u f(u, v, w, \dots) = f(qu, v, w, \dots). \qquad \dots (3.1)$$

For the generalized $q$-Appell function, we can easily verify the following recurrence relations :

$$\left(1 - a\,T_x T_y^{d}\right) \mathscr{F}(a) = (1-a)\,\mathscr{F}(aq)$$

$$\left(1 - a'\,T_y^{d'}\right) \mathscr{F}(a') = (1-a')\,\mathscr{F}(a'q)$$

$$\left(1 - bT_x\, T_y^{e}\right) \mathscr{F}(b) = (1-b)\,\mathscr{F}(bq)$$

$$\left(1 - b'\, T_y^{e'}\right) \mathscr{F}(b') = (1-b')\,\mathscr{F}(b'\,q)$$

$$\left(1 - cq^{-1}\, T_x\, T_y^{f}\right) \mathscr{F}(c) = (1-cq^{-1})\,\mathscr{F}(cq^{-1})$$

$$\left(1 - c'\,q^{-1}\, T_y^{f'}\right) \mathscr{F}(c') = (1-c'\,q^{-1})\,\mathscr{F}(c'\,q^{-1})$$

$$x^{-1}(1-T_x)\,\mathscr{F}\left(\begin{matrix} a, b \\ c \end{matrix}\right) = \frac{(1-a)(1-b)}{(1-c)}\,\mathscr{F}\left(\begin{matrix} aq,\ bq \\ cq \end{matrix}\right)$$

$$y^{-1}(1-T_y)\,\mathscr{F}\left(\begin{matrix} a, a', b, b' \\ c, c' \end{matrix}\right) = \frac{(a;q)_d\,(a';q)_{d'}\,(b;q)_e\,(b';q)_{e'}}{(c;q)_f\,(c';q)_{f'}}$$

$$\times\ \mathscr{F}\left(\begin{matrix} aq^{d}, & a'q^{d'}, dq^{e}, b'q^{e'} \\ cq^{f}, & c'q^{f'} \end{matrix}\right). \qquad \ldots(3.2)$$

The relations (3.2) imply the fundamental difference equations satisfied by $\mathscr{F}$ :

$$\left[x\left(1 - a\,T_x\,T_y^{d}\right)\left(1 - bT_x\,T_y^{e}\right) - (1-T_x)\left(1 - cq^{-1}T_x\,T_y^{f}\right)\right]$$
$$\mathscr{F} = 0 \qquad \ldots(3.3)$$

and

$$\left[\left\{y\left(1 - aT_x\,T_y^{d}\right)^{d}\left(1 - a'\,T_y^{d'}\right)^{d'}\left(1 - bT_x\,T_y^{e}\right)^{e}\right.\right.$$
$$\times\left(1 - b'\,T_y^{e'}\right)^{e'}\Big\} - \Big\{(1-T_y)\left(1 - cq^{-1}\,T_x\,T_y^{f}\right)^{f}$$
$$\left.\left.\times\left(1 - c'\,q^{-1}\,T_y^{f'}\right)^{f'}\right\}\right]\mathscr{F} = 0. \qquad \ldots(3.4)$$

To cast the system into a canonical form, we define a new function $\widetilde{\mathscr{F}}$ of eight variables $u_1, u_2\ \ldots, u_8$.

$$\widetilde{\mathscr{F}} = \mathscr{F}\left(\begin{matrix} a, a'\ b, b'; \\ c, c' \end{matrix}\ x, y\right) u_1^{-\alpha}\, u_2^{-\alpha'}\, u_3^{-\beta}\, u_4^{-\beta'}\, u_5^{\gamma-1}\, u_6^{\gamma'-1}$$

$$\text{where } x = \frac{u_5\,u_7}{u_1\,u_3} \text{ and } y = \frac{u_5^{f}\,u_6^{f'}\,u_8}{u_1^{d}\,u_2^{d'}\,u_3^{e}\,u_4^{e'}} \qquad \ldots(3.5)$$

and as defined earlier, $a = q^{\alpha}$, $a' = q^{\alpha'}$ and so on.

Let $\Delta_p^+$ and $\Delta_p^-$ be the $q$-difference operators

$$\Delta_p^+ = u_p^{-1}\left(1 - T_{u_p}\right)$$

$$\Delta_p^- = u_p^{-1}\left(1 - T_{u_p}^{-1}\right). \qquad \ldots(3.6)$$

In terms of these operators, relations (3.2) take the simple form

$$\Delta_1^- \tilde{\mathscr{F}}(a) = (1-a)\,\tilde{\mathscr{F}}(aq)$$

$$\Delta_2^- \tilde{\mathscr{F}}(a') = (1-a')\,\tilde{\mathscr{F}}(a'q)$$

$$\Delta_3^- \tilde{\mathscr{F}}(b) = (1-b)\,\tilde{\mathscr{F}}(bq)$$

$$\Delta_4^- \tilde{\mathscr{F}}(b') = (1-b')\,\tilde{\mathscr{F}}(b'q)$$

$$\Delta_5^+ \tilde{\mathscr{F}}(c) = (1-cq^{-1})\,\tilde{\mathscr{F}}(c\,q^{-1})$$

$$\Delta_6^+ \tilde{\mathscr{F}}(c') = (1-c'\,q^{-1})\,\tilde{\mathscr{F}}(c'\,q^{-1})$$

$$\Delta_7^+ \tilde{\mathscr{F}}\begin{pmatrix} a, b \\ c \end{pmatrix} = \frac{(1-a)(1-b)}{(1-c)}\,\tilde{\mathscr{F}}\begin{pmatrix} aq, bq \\ cq \end{pmatrix}$$

$$\Delta_8^+ \tilde{\mathscr{F}}\begin{pmatrix} a, a'\, b, b' \\ c, c' \end{pmatrix} = \frac{(a;q)_d\,(a';q)_{d'}\,(b;q)_e\,(b';q)_{e'}}{(c;q)_f\,(c';q)_{f'}}$$

$$\times\,\tilde{\mathscr{F}}\begin{pmatrix} aq^d,\ a'\,q^{d'},\ bq^e,\ b'q^{e'} \\ cq^f,\ c'q^{f'} \end{pmatrix} \qquad \ldots(3.7)$$

where relations of the type

$\Delta_1 \tilde{\mathscr{F}}(a) = (1-a)\,\tilde{\mathscr{F}}(aq)$ implies that the parameter $a$ is transformed to $aq$, that is $\alpha$ is transformed to $\alpha + 1$ in $\tilde{\mathscr{F}}$ (since $a = q^{\alpha}$) and other parameters are unaffected. (3.3) and (3.4) will become the following canonical partial $q$-difference equations satisfied by $\tilde{\mathscr{F}}$:

$$\left(\Delta_1^- \Delta_3^- - \Delta_5^+ \Delta_7^+\right) \tilde{\mathscr{F}} = 0$$

and

$$\left(\Delta_1^{-d}\,\Delta_2^{-d'}\,\Delta_3^{-e}\,\Delta_4^{-e'} - \Delta_5^{+f}\,\Delta_6^{+f'}\,\Delta_8^{+}\right)\tilde{\mathscr{F}} = 0. \qquad \ldots(3.8)$$

Furthermore $\tilde{\mathscr{F}}$ satisfies the dilation eigenvalue equations

$$\left.\begin{array}{ll} T_1\,T_7\,T_8^{d} \sim a^{-1}, & T_2\,T_8^{d'} \sim a'^{-1} \\ T_3\,T_7\,T_8^{e} \sim b^{-1}, & T_4\,T_8^{e'} \sim b'^{-1} \\ T_5\,T_7^{-1}\,T_8^{-f} \sim cq^{-1}, & T_6\,T_8^{-f'} \sim c'q^{-1} \end{array}\right\} \qquad \ldots(3.9)$$

where $T_p$ stands for $T_{u_p}$ and $A \sim a$, where $A$ is a $q$-difference operator and $a \in ¢$, implies $A\,\tilde{\mathscr{F}} = a\,\tilde{\mathscr{F}}$.

Indeed, $\tilde{\mathscr{F}}$ is characterized by (3.8) and (3.9). It is (to within a constant multiple) the only solution of these equations analytic in $u_1, u_2, \ldots, u_8$ at $u_7 = u_8 = 0$. We can regard an analytic solution of the canonical equations (3.8) as a generating function for generalized $q$-Appell functions, following Weisner's method[7]. It is also relevant to mention here that the canonical equation

$$\Delta_1^{-}\,\Delta_3^{-} - \Delta_5^{+}\,\Delta_7^{+} \sim 0$$

is the $q$-analog of a wave equation in four variables.

The results for each of the $q$-Appell functions $f_1, f_2, f_3$ and $f_4$ are presented in the following table :

| Function | Canonical equations | Eigen value equations | Eigen functions |
|---|---|---|---|
| 1. $\mathscr{F}$ | $\Delta_1^{-}\,\Delta_3^{-} - \Delta_5^{+}\,\Delta_7^{+} \sim 0$ | $T_1\,T_7\,T_8^{f} \sim a^{-1}$ | $\mathscr{F}\left(\begin{matrix} a, a', b, b'; \\ c, c'; \end{matrix}\; x, y\right)$ |
| | $\Delta_1^{-d}\Delta_2^{-d'}\,\Delta_3^{-e}\,\Delta_4^{-e'}$ | $T_2\,T_8^{d'} \sim a'^{-1}$ | $x u_1^{-\alpha} u_{u_2}^{-\alpha'} u_3^{-\beta} u_4^{-\beta'} u_5^{\gamma-1} u_6^{\gamma' 1}$ |
| | $-\ \Delta_5^{+f}\Delta_6^{+f'}\,\Delta_8 \sim 0$ | $T_3\,T_7\,T_8^{e} \sim b^{-1}$ | where |
| | | $T_4 T_8^{e'} \sim b'^{-1}$ | $x = \dfrac{u_5 u_7}{u_2\,u_3}$ |
| | | $T_5 T_7^{-1}\,T_8^{-f} \sim cq^{-1}$ | and |
| | | $T_6 T_8^{-f'} \sim c'\,q^{-1}$ | $y = \dfrac{u_5^{f}\,u_6^{f'}\,u_8}{u_1^{d}\,u_2^{d'}\,u_3^{e}\,u_3^{e}\,u_4^{e'}}$ |

(equation continued on p. 982)

| Function | Canonical equations | Eigen value equations | Eigen functions |
|---|---|---|---|
| 2. $f_1$ | $\Delta_1^- \Delta_3^- - \Delta_5^+ \Delta_7^+ \sim 0$ | $T_1 T_7 T_8 \sim a^{-1}$ | |
| | $\Delta_1^- \Delta_4^- - \Delta_5^+ \Delta_8^+ \sim 0$ | $T_3 T_7 \sim b^{-1}$ | $f_1\left(\begin{matrix} a, b, b'; \\ c; \end{matrix} \frac{u_5 u_7}{u_1 u_3}, \frac{u_5 u_8}{u_1 u_4}\right)$ |
| | | $T_4 T_8 \sim b'^{-1}$ | |
| | | $T_5 T_7^{-1} T_8^{-1} \sim cq^{-1}$ | $x u_1^{-\alpha} u_2^{-\beta} u_4^{-\beta'} u_5^{\gamma-1}$ |
| 3. $f_2$ | $\Delta_1^- \Delta_3^- - \Delta_5^+ \Delta_7^+ \sim 0$ | $T_1 T_7 T_8 \sim a^{\;1}$ | $f_2\left(\begin{matrix} a, b, b'; \\ c, c'; \end{matrix} \frac{u_5 u_7}{u_1 u_3}, \frac{n_6 u_8}{u_1 u_4}\right)$ |
| | $\Delta_1^- \Delta_4^- - \Delta_6^+ \Delta_8^+ \sim 0$ | $T_3 T_7 \sim b^{-1}$ | |
| | | $T_4 T_8 \sim b'^{-1}$ | $\times u_1^{-\alpha} u_3^{-\beta} u_4^{-\beta'} x_5^{\gamma\;1} u^{\gamma'-1}$ |
| | | $T_5 T_7^{-1} \sim cq^{-1}$ | |
| | | $T_6 T_8^{-1} \sim c' q^{-1}$ | |
| 4. $f_3$ | $\Delta_1^- \Delta_3^- - \Delta_5^+ \Delta_7^+ \sim 0$ | $T_1 T_7 \sim a^{-1}$ | $f_3\left(\begin{matrix} a, a', b, b'; \\ c, c'; \end{matrix} \frac{u_5 u_7}{u_1 u_3}, \frac{u_5 u_8}{u_2 u_4}\right)$ |
| | $\Delta_2^- \Delta_4^- - \Delta_5^+ \Delta_8^+ \sim 0$ | $T_2 T_8 \sim a'^{\;1}$ | |
| | | $T_3 T_7 \sim b^{-1}$ | |
| | | $T_4 T_8 \sim b'^{-1}$ | $\times u_1^{-\alpha} u_2^{-\alpha'} u_3^{-\beta} u^{-\beta'} u_5^{\gamma-1}$ |
| | | $T_5 T_7^{-1} T_8^{-1} \sim cq^{-1}$ | |
| 5. $f_4$ | $\Delta_1^- \Delta_3^- - \Delta_5^+ \Delta_7^+ \sim 0$ | $T_1 T_7 T_8 \sim a^{-1}$ | $f_4\left(\begin{matrix} a, b; \\ c, c': \end{matrix} \frac{u_5 u_7}{u_1 u_3}, \frac{u_6 u_8}{u_1 u_3}\right)$ |
| | $\Delta_1^- \Delta_3^- - \Delta_6^+ \Delta_8^+ \sim 0$ | $T_3 T_7 T_8 \sim b^{-1}$ | |
| | | $T_5 T_7^{-1} \sim cq^{-1}$ | |
| | | $T_6 T_8^{-1} \sim c' q^{-1}$ | $\times u_1^{-\alpha} u_3^{-\beta} u_5^{\gamma-1} u_6^{\gamma'-1}$ |

## 4. Generating Functions for $q$-Appell Functions

The smooth extension of Lie algebraic theory to local Lie group theory and obtaining generating functions thereby, is not applicable here. There is no $q$-analogy to exponentiation of differential operators in the ordinary theory. However, in particular cases, the analogy is successful. Here we will present some examples showing how generating functions for the $q$-Appell functions are associated with the canonical equations (3.8). Here it is usefult o recall that $\widetilde{\mathscr{F}}$ given by (3.5) is a solution of (3.8), and that $\Delta_1^-$ , $\Delta_2^-$ , $\Delta_3^-$, $\Delta_4^-$ $\Delta_5^+$ , $\Delta_6^+$ , $\Delta_7^+$ , $\Delta_8^+$ and the dilation operators in (3.9) are symmetries of equations (3.8). Here again we follow the approach of Agarwal *et al.*[1].

Consider the $q$-exponential

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \quad |x| < 1 \qquad ..(4.1)$$

satisfying $\Delta_x^+ e_q = e_q$ (Slater[6], p. 92).

In a formal sense atleast, we can say that the operator $e_q\left(\lambda\ \Delta_1^-\right)$, $\lambda \in ¢$. is a symmetry of (3.8). Applying this operator to a basis solution $\widetilde{\mathscr{F}}$ (3.5), and making use of (3.7) and (4.1) we obtain,

$$e_q\left(\lambda\ \Delta_1^{-1}\right)\widetilde{\mathscr{F}}(a) = \sum_{n=9}^{\infty} \lambda^n \frac{(a\,;q)_n}{(q\,;q)_n}\widetilde{\mathscr{F}}(aq^n). \qquad ...(4.2)$$

To calculate the left hand side of (4.2), we employ Heine's ($q$-binomial) Theorem (Slater[6], page 92)

$$\sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} t^r = \frac{(at\,;q)_\infty}{(t\,;q)_\infty}. \qquad ..(4.3)$$

Using (4.3) we derive,

$$e_q\left(\lambda\ \Delta_x^-\right)x^n = x^n \frac{(\lambda q^{-n}/x\,;q)_\infty}{(\lambda/x;q)_\infty} = \frac{x^n}{(\lambda/x,q)_{-n}}. \qquad ..(4.4)$$

From (4.4), (2.2) and (3.5) we get

$$e_q\left(\lambda\ \Delta_1^-\right)\widetilde{\mathscr{F}} =$$

$$\frac{(\lambda a/u_1;q)_\infty}{(\lambda/u_1;q)_\infty}\left(\sum_{m\cdot n=0}^{\infty} \frac{(a;q)_{m+dn}\,(a';q)_{d'n}\,(b;q)_{m+en}\,(b';q)_{e'n}\,x^m y^n}{(\lambda a/u_1\,;q)_{m+dn}\,(c\,;q)_{m+fn}\,(c'\,;q)_{f'n}\,(q;q)_m\,(q;q)_n}\right)$$

(*equation continued on p. 984*)

$$\times\ u_1^{-\alpha}\ u_2^{-\alpha'}\ u_3^{-\beta}\ u_4^{-\beta'}\ u_5^{\gamma-1}\ u_6^{\gamma'-1} \qquad \ldots(4.5)$$

where

$$x = \frac{u_5\ u_7}{u_1\ u_3} \text{ and } y = \frac{u_5^{f}\ u_6^{f'}\ u_8}{u_1^{d}\ u_2^{d'}\ u_3^{e}\ u_4^{e'}}\ .$$

Equating (4.2) and (4.5) and writing $\lambda/u_1$ as $t$, we get

$$\frac{(ta;\ q)_\infty}{(t\ :\ q)_\infty} \times \sum_{m\cdot n=0}^{\infty} \frac{(a;\ q)_{m+nd}\ (a';\ q)_{d'n}\ (b;\ q)_{m+en}\ (b';\ q)_{e'n}\ x^m\ y^n}{(at;\ q)_{m+dn}\ (c;\ q)_{m+fn}\ (c;\ q)_{f'n}\ (q;\ q)_m\ (q;\ q)_n}$$

$$= \sum_{r=0}^{\infty} \frac{(a;\ q)_r}{q;\ q)_r}\ t^r\ \mathscr{F}\left(\begin{matrix} aq^r,\ b,\ b';\\ c,\ c':\end{matrix}\ x,\ y\right). \qquad \ldots(4.6)$$

No doubt, the generating relation (4.6) does not look very elegant. The result looks cumbersome because of our idea of developing a unified approach to all the four basic Appell functions. When we consider (4.6) in individual context, the relation will look much more elegant and meaningf ul since the left hand side functions in (4.6) will be $q$-analogs of Kampe' de Fériet functions. Furthermore, in this case as well as in all the results later obtained in this paper we can deduce the results for ordinary Appell functions by making $q \to 1$ while noting down $a = q^\alpha$ and so on. Similar generating relations for $\mathscr{F}$ can be obtained by taking operators $\Delta_2^-$, $\Delta_3^-$ and $\Delta_4^-$ and proceeding in the same lines as above.

To obtain series identities concerning $q$-exponential of $\Delta^+$ operators we have to consider the other $q$-exponentiat[6],

$$E_q\ (x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q\ ;\ q)_n}\ x^n = (-x\ ;\ q)_\infty\ . \qquad \ldots(4.7)$$

Like in the previous case, we can consider $E_q\left(-\lambda\ \Delta_5^+\right)$ and $E_q\left(-\lambda\ \Delta_7^+\right)$ to be symmetries of (3.8).

It is also obvious that

$$\Delta^-{}_x\ E_q = -\ q^{-1}\ E_q$$

and $e_q\ (x)\ E_q\ (-x) = 1$ (Slater[6], p. 92).

We can verifiy using $q$-binomial theorem that

$$E_q\ (-\ \lambda\Delta_x^+\ )\ x^n = x^n\ \frac{(\lambda/x;\ q)_\infty}{(\lambda q^n/x;\ q)_\infty}\ . \qquad \ldots(4.8)$$

We have from (3.7) and (4.7)

$$E_q\left(-\ \lambda\,\Delta_5^+\right)\tilde{\mathscr{F}}(c) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n}\,(-\lambda)^n\,(cq^{-1};q)_n\,\tilde{\mathscr{F}}(cq^{-n})$$

$$= \sum_{n=0}^{\infty} (\lambda\,c/q)^n\,\frac{(c^{-1}q;q)_n}{(q;q)_n}\,\tilde{\mathscr{F}}(cq^{-n}) \qquad \ldots(4.9)$$

and

$$E_q\left(-\lambda\,\Delta_7^+\right)\tilde{\mathscr{F}}\begin{pmatrix} a,\,b \\ c \end{pmatrix} = \sum_{n=0}^{\infty} q^{n(n-1)/2}\,\frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(q;q)_n}\,(-\lambda)^n$$

$$\times\ \tilde{\mathscr{F}}\begin{pmatrix} aq^n,\ bq^n \\ cq^n \end{pmatrix}. \qquad \ldots(4.10)$$

As earlier we can calculate the left hand side of (4.9) and (4.10) using (4.8) and (3.5). After some simplification, we get the following generating relations :

$$\frac{(qt/c;\,q)}{(t;\,q)} \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+nd}\,(a';q)_{n'd}\,(b;q)_{m+en}\,(b';q)_{e'n}\,(t;q)_{m+nf}\,x^m\,y^n}{(c;\ q)_{m+nf}\,(c';q)_{f'n}\,(q;q)_m\,(q;q)_n}$$

$$= \sum_{r=0}^{\infty} \frac{(q/c;q)_r}{q\ ;\,q)_r} \times \mathscr{F}\left( \frac{a,\,a',\,b,\,b';}{cq^{-r},\,c';}\ x,\,y \right) t^r \qquad \ldots(4.11)$$

and

$$\sum_{m,n=0}^{\infty} \frac{(a;q)_{m+nd}\,(a';q)_{n'd}\,(b;q)_{m+ne}\,(b';q)_{n'e}\,(t;q)_m\,x^m\,y^n}{(c;q)_{m+nf}\,(c';q)_{n'f}\,(q;q)_m\,(q;q)_n}$$

$$= \sum_{r=0}^{\infty} q^{r(r-1)/2}\,\frac{(a;q)\,(b;q)_r}{(c;q)_r\,(q;q)_r}\,\mathscr{F}\begin{pmatrix} aq^r,\,a',\,bq^r,\,b'; & x,\,y \\ cq^r,\,c'; & \end{pmatrix}$$

$$(-tx)^r. \qquad \ldots(4.12)$$

Again (4.11) and (4.12) will give more manageable and elegant results when we consider the individual cases, that is taking any one set of values from (2.3).

Now we are going to consider some examples where the generating functions are directly characterized in terms of symmetry operators of the canonical equations (3.8).

As a very simple we aim at an eigen function characterized by the following relations :

$$\left.\begin{aligned}&\Delta_1^- \sim 1, \quad T_2 T_8^{d'} \sim a'^{-1}\\&T_3 T_7 T_8^{e} \sim b^{-1}, \quad T_4 T_8^{e'} \sim b'^{-1}\\&T_5 \; T_7^{-1} \; T_8^{-f} \sim cq^{-1}, \; T_6 T_8^{f'} \sim c' q^{-1};\end{aligned}\right\} \qquad \ldots(4.13)$$

These equations with $\Delta_1$ replaced by 1 in (3.8) are in canonical form. In this case the variables separate and we see that the function

$$E_q(-u_1 q)\,\mathscr{F}\left(\begin{matrix}0, a', b, b'; \\ c, c';\end{matrix}\; \frac{u_5 u_7}{u_3} \; \frac{u_5^{f}\; u_6^{f'}\; u_8}{u_2^{d'}\; u_3^{e}\; u_4^{e'}}\right)$$

$$u_2^{-\alpha'}\; u_3^{-\beta}\; u_4^{-\beta'}\; u_5^{\gamma-1}\; u_6^{\gamma'-1} \qquad \ldots(4.14)$$

satisfies these equations.

The following generating relation is an immediate consequence:

$$(tq;\, q)_\infty\, \mathscr{F}\left(\begin{matrix}0, a', b, b'; \\ c, c';\end{matrix}\; xt,\, yt^d\right)$$

$$= \sum_{n=0}^{\infty} k_n\, \mathscr{F}\left(\begin{matrix}q^{-n}, a', b, b'; \\ c, c';\end{matrix}\; x, y\right) t^n. \qquad \ldots(4.15)$$

Setting $x = y = 0$ and using (4.7) we get that

$$k_n = \frac{(-1)^n\, q^{n(n+1)/2}}{(q;\, q)_n}\,. \qquad \ldots(4.16)$$

By considering other $\Delta^-$ and $\Delta^+$ operators we can derive similar series identities.

Consider another example :

$$\Delta_1\, \Delta_3 \sim 1, \qquad T_1\; T_3^{-1} \sim a$$

$$T_2\; T_8^{d'} \sim a'^{-1},\; T_4\; T_8^{e'} \sim b'^{-1}$$

$$T_5\; T_7^{-1}\; T_8^{-f} \sim cq^{-1},\; T_6\, T_8^{f'} \sim c' q^{-1} \qquad \ldots(4.17)$$

together with the canonical equations (3.8).

Here the calculations will be tedious if we consider all values for $d$, $d'$, $e$, $e'$, $f$ and $f'$. Hence, we consider two individual sets of values from (2.3) for which we get generating identities without much difficulty.

First let

$$(d, d', e, e', f, f') = (1, 0, 1, 0, 0, 1). \qquad \ldots(4.18)$$

We see $\mathscr{F} = f_4$ for these values from (2.3). We see that the function

$$ {}_0\hat{\phi}_1 \left( \begin{matrix} - \\ aq^{-1} \end{matrix} ; u_1 u_3 \right) f_4 \left( \begin{matrix} 0, 0; \\ c, c'; \end{matrix} u_5 u_7, u_6 u_8 \right) u_1^{-\alpha} u_5^{\gamma-1} u_6^{\gamma'-1} \qquad \ldots(4.19)$$

where

$$ {}_0\hat{\phi}_1 \left( \begin{matrix} 0 \\ a \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{x^n}{(a; q^{-1})_n (q^{-1}; q^{-1})_n} \qquad \ldots(4.20)$$

satisfies (4.17) together with (3.8) for the set of values (4.18)

Therefore, we get the following generating relation :

$$ {}_0\hat{\phi}_1 \left( \begin{matrix} - \\ aq^{-1} \end{matrix} ; t \right) f_4 \left( \begin{matrix} 0, 0; \\ c, c'; \end{matrix} xt, yt \right)$$

$$= \sum_{n=0}^{\infty} c_n \, f_4 \left( \begin{matrix} aq^{-n}, q^{-n}; \\ c, c'; \end{matrix} x, y \right) t^n. \qquad \ldots(4.21)$$

Setting $x = y = 0$ we get

$$c_n = \frac{1}{(aq^{-1}; q^{-1})_n (q^{-1}; q^{-1})_n}. \qquad \ldots(4.22)$$

Next let

$$(d, d', e, e', f, f') = (0, 1, 0, 1, 1, 0). \qquad \ldots(4.23)$$

For these values $\mathscr{F} = f_3$.

Proceeding as in the previous case for this set of values (4.23) we get the generating relation

$$ {}_0\hat{\phi}_1 \left( \begin{matrix} - \\ aq^{-1} \end{matrix} ; t \right) f_3 \left( \begin{matrix} 0, a', 0, b'; \\ c; \end{matrix} xt, y \right)$$

$$= \sum_{n=0}^{\infty} c_n \, f_3 \left( \begin{matrix} aq^{-n}, a', q^{-n}, b'; \\ c; \end{matrix} x, y \right) t^n. \qquad \ldots(4.24)$$

We see that $c_n$ is again given by (4.22).

## Conclusion

A detailed study of $q$-Appell functions through Lie theory has been made. In the process new series identities were evolved. We can proceed in the same way to $q$-Lauricella functions and get the corresponding results. Also results for ordinary hypergeometric functions can be reduced by making $q \to 1$.

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# ON DISTRIBUTIONAL LAPLACE-HARDY $\mathscr{L} F_\nu$ TRANSFORMATION

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The aim of the present paper is to extend Classical Laplace-Hardy $LF_\nu$ transform to generalized functions. Analyticity theorem, order property, boundedness theorem, inversion theorem, uniqueness theorem, representation theorem and operational transform formulae for Laplace-Hardy $\mathscr{L}F_\nu$ transformation of generalized functions have been proved.

## 1. Introduction

The classical Hardy's $C_\nu$-transforms and $F_\nu$-transforms with their inversion formulae are represented by the following two integral equations[4] (p. 694):

$$f(x) = \int_0^\infty F_\nu(tx)\, tdt \int_0^\infty C_\nu(ty)\, yf(y)\, dy \qquad \ldots(1.1)$$

and

$$f(x) = \int_0^\infty C_\nu(tx)\, tdt \int_0^\infty F_\nu(ty)\, y f(y)\, dy \qquad \ldots(1.2)$$

where

(a) $C_\nu(z) = \cos(p\pi) J_\nu(z) + \sin(p\pi) Y_\nu(z)$

(b) $$F_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+\nu+1)}$$

$$= \frac{2^{2-\nu-2p}\, S_{\nu+2p-1,\nu}(z)}{\Gamma(p)\,\Gamma(\nu+p)}$$

(c) $J_\nu(z)$ and $Y_\nu(z)$ are bessel's functions of first kind and second kind[3] (p. 344) respectively, and

(d) $S_{\mu,\nu}(z)$ being Lommel's function[8] (p. 345), $\mu = \nu + 2p - 1$.

The kernel function $F_\nu$ in (1.1) is the solution of Bessel's differential equation

$$\beta_{\nu,x}(F_\nu(x)) = x^{\nu+2p-2}\ t^{\nu+2p}$$

where

$$\beta_{\nu,x} = \left( D_x^2 + \frac{1}{x} - \frac{\nu^2}{x^2} \right)$$

(Pathak and Pandey, p. 760).

The classical two sided Laplace transform is given by

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt, \ \operatorname{Re} s > 0 \qquad \ldots(1.3)$$

where $f(t)$ is suitably restricted conventional function on the real line $-\infty < t < \infty$.

The conventional Laplace-Hardy $LC_\nu$ transform of a complex valued smooth function $\phi(t, x)$ is defined by the convergent integral,

$$F(s, y) = LC_\nu(\phi(t, x)) = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-st}\, C_\nu(x\,y)\, \phi(t, x)\, xd\, xdt \qquad \ldots(1.4)$$

where $a$ and $b$ are real numbers such that $a + \operatorname{Re} s > 0$ and $b + \operatorname{Re} s < 0$, $s = \sigma + iw$, $|\nu| \leqslant \alpha \leqslant \frac{1}{2}$. The inversion formula for (1.3) due to authors[1] is given below :

$$\phi(t, x) = \lim_{r \to \infty} (2\pi i)^{-1} \int_{0}^{\infty} \int_{\sigma-ir}^{\sigma+ir} e^{st}\, F(s, y)\, F_\nu(xy)\, y\, dy\, ds \qquad \ldots(1.5)$$

where $r$ is a real number and $\sigma$ is any fixed real number such that $\sigma_1 < \sigma < \sigma_2$ with $\operatorname{Re} s = \sigma$ (Zemanian, pp. 35, 36).

The conventional Laplace-Hardy $LC_\nu$ transform (1.3) has recently been extended to a certain class of generalized functions by authors[1].

Throughout the paper the proof of standard lemmas and theorems are clear and they are omitted.

## 2. Testing Function Space $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a\,b}(\Omega)$

Let $a, b, t \in R^1$, $s \in C^1$ and for each $j = 0, 1, 2, \ldots$ $\eta_{a,b,j}(t, x)$ be the function such that

$$\eta_{a,b,j}(t, x) = \begin{cases} e^{at}\, x^{-2j+\alpha} & 0 \leqslant t < \infty;\ 0 < x \leqslant 1 \\ e^{bt}\, x^{-\beta-2} & -\infty < t < 0;\ x > 1 \end{cases} \qquad \ldots(2.1)$$

where $\alpha$ and $\beta$ are fixed numbers satisfying $\alpha+\nu+2p \geqslant +0$, $\beta \geqslant \sigma = \max(\nu + 2p-2, -1/2)$ respectively for $-1/2 \leqslant \nu \leqslant 1/2$ and real $p$.

Now for each non-negative integer $K = (k_1, k_2)$ belonging to $R^2$, we define a space $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ consisting of all infinitely differentiable functions $\phi(t, x)$ defined over the domain,

$$\Omega = \{(t, x) \mid -\infty < t < \infty;\ 0 < x < \infty\}$$

satisfying

$$\rho^{\alpha,\beta}_{a\cdot b\cdot k} \quad (\phi(t, x)) = \sup_{\substack{-\infty < t < \infty \\ 0 < x < \infty}} \left| \eta_{a,b,j}(t,x)\ D_t^{k_1}\ \beta^{k_2}_{\nu,x} \left( \frac{\phi(t, x)}{x} \right) \right| < \infty \qquad \ldots(2.2)$$

where $\beta^{k_2}_{\nu,x}$ is Bessel's differential operator defined in Section 1, and $D_x = \dfrac{\partial}{\partial x}$ $D_t = \dfrac{\partial}{\partial t}$. Obviously $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ is a linear space under the pointwise addition of functions and their multiplication by complex number. Clearly $\rho^{\alpha,\beta}_{a\cdot b\cdot k}$ is a norm for $K = 0$. Therefore the collection of seminorms $\left\{\rho^{\alpha,\beta}_{a\cdot b\cdot k}\right\}^{\infty}_{k_1 k_2=0}$ is a multinorm for $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ and we assign to $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ the topology[1] generated by the countable multinorm $\left\{\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}\right\}^{\infty}_{k_1 k_2=0}$.

Thus the linear space $\mathscr{L} H^{\nu,p}_{\sigma\cdot\beta\cdot a\cdot b}$ $(\Omega)$ with the topology generated by

$$\left\{\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}\right\}^{\infty}_{k_1 k_2=0}$$

is a countabe multinorm space.

We say that the sequence $\{\phi m\,(t, x)\}^{\infty}_{m=1}$ in $\mathscr{L} H^{\nu,p}_{\alpha,\beta\cdot a\cdot b}$ $(\Omega)$ converges to $\phi\,(t, x)$ in $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ if for each fixed $a$, $b$, $K$, $\alpha$ and $\beta$, $\rho^{\alpha,\beta}_{a\cdot b\cdot k}$ $(\phi m - \phi)$ tends to zero as $m$ tends to $\infty$. Similarly we say that the sequence $\{\phi m\,(t, x)\}^{\infty}_{m=1}$ in $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ is a Cauchy sequence if for each fixed $a$, $b$, $k$, $\alpha$ and $\beta$, $\rho^{\alpha,\beta}_{a\cdot b\cdot k}$ $(\phi m - \phi n)$ tends to zero as $m$ and $n$ tend to $\infty$ independently.

*Lemma* 2.1— $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ is complete and therefore a Frechet space.

*Lemma* 2.2—$\mathscr{L} H^{\nu,p}_{\alpha,\beta\cdot a\cdot b}$ $(\Omega)$ is a testing function space.

## The Countable-union Space and Its Dual $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z)$

We shall now turn to certain countable union space $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z)$ generated from $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b}(\Omega)$. Let $\omega$ denote either a finite number or $-\infty$ and $z$ denote a a finite number or $+\infty$. Consider two monotonic sequences of real numbers $\{am\}_{m=1}^{\infty}$ and $\{bm\}_{m=1}^{\infty}$ such that $am \to \omega_+$ and $bm \to z_-$.

Let $\left\{\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m}\right\}_{m=1}^{\infty}$ be a sequence of countably multinormed spaces such that

$$\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_1,b_1} \subset \mathscr{L}H^{\nu,p}_{\alpha,\beta,a_2,b_2} \subset \ldots \subset \mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m}$$

$$\subset \mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m+1,b_m+1} \subset \ldots$$

where $am < am_{+1}$, $bm < bm_{+1}$ ....

Further assume that the topology of each $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m}$ is stronger than the topology induced on it by $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_{m+1},b_{m+1}}$. Let $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega,z)$ denote the union of all these spaces. Thus

$$\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z) = \bigcup_{m=1}^{\infty} \mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m} \qquad \ldots(2.4)$$

A sequence $\{\phi m\}_{m=1}^{\infty}$ converges in $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z)$ if and only if it converges in one of the $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m}(\Omega)$ spaces and this definition does not depend on the choices of $\{am\}$ and $\{bm\}$. Since for each $m$, $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a_m,b_m}(\Omega)$ is complete and hence the countable union space $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z)$ is also complete. $(\mathscr{L}H^{\nu,p}_{\alpha,\beta})'(\omega, z)$ denotes the dual space of $\mathscr{L}H^{\nu,p}_{\alpha,\beta}(\omega, z)$. $(\mathscr{L}H^{\nu,p}_{\alpha,\beta})'(\omega, z)$ is also complete (Zemanian[9], Theorem 1.9.2).

Now we give some lemmas which will be found useful in the sequel.

*Lemma* 2.3—If $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$, $p$ real $\alpha$ and $\beta$ are fixed real numbers satisfying $\alpha + \nu + 2p \geqslant 0$, $\beta \geqslant \eta = \max(\nu + 2p - 2, -\frac{1}{2})$ then for positive fixed values of $y$ and Re $s > 0$, $e^{-st}\, x\, F_\nu(x\, y)$ belongs to $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b}(\Omega)$ as a function of $x$ and $t$ where $F_\nu(xy)$ is the same as defined in the article Section 1.

PROOF : Consider,

$$\rho^{\alpha,\beta}_{a,b,k}(e^{-st}\, xF_\nu(xy)) = \operatorname*{Sup}_{\Omega} \left| \eta_{a,b,j}(t,x)\; D_t^{k_1}\; \beta^{k_2}_{\nu,x}\; x\left(\frac{e^{-st}\, xF_\nu(xy)}{x}\right)\right|$$

$$= \operatorname*{Sup}_{\Omega} \left| \eta_{a,b,j}(t,x)\,(-1)^{k_1} s^{k_1} e^{-st}\left\{(-1)^{k_2} y^{2k_2} F_\nu(xy) - y^{\nu+2p} \sum_{i=1}^{k_2} a_i\, x^{\nu+2p-2i}\, y^{2k_2-2i}\right\}\right|$$

$$\leqslant \operatorname*{Sup} \left| \eta_{a,b,j}(t,x)\, x^{\nu+2p}\, y^{\nu+2p+k_2} s^{k_1} e^{-st}\left\{(xy)^{-(\nu+2p)} F_\nu(xy) + \sum_{i=1}^{k_2} a_i (xy)^{-2i}\right\}\right|.$$

Now by using asymptotic orders of $F_\nu(xy)$ Pathak and Pandey[6] (p. 96) it is easy to see that $|(xy)^{-(\nu+2p)} F_\nu(xy)|$ is bounded by $M$ for $x > 0$, fixed by $y > 0$ and the series on the right hand side in the bracket is a series of positive finite terms which is bounded by $M_1$ for $x > 0$, fixed $y > 0$ and $a_{k_2}$ being the certain constants depending on $\nu$ and $p$.

Therefore

$$\rho^{\alpha,\beta}_{a,b,k}(e^{-st}\, x F_\nu\, xy)) = \operatorname*{Sup}_{\Omega} |\eta_{a,b,j}(t,x)\, x^{\nu+2p}\, e^{-st}(M+M_1)||s^{k_1}\, y^{\nu+2p\neq 2k2}|.$$

Now $\operatorname*{Sup}_{\Omega} |\eta_{a,b,j}(t,x)\, x^{\nu+2p}\, e^{-st}|$ is bounded for $\alpha + \nu + 2p \geqslant 0$,

$|\nu| \leqslant \alpha \leqslant \frac{1}{2}$, $\beta \geqslant \sigma = \max(\nu + 2p-2, -\frac{1}{2})$ and $a < \operatorname{Re} s < b$.

$|s^{k_1}|\;|Y^{\nu+2p+2k_2}|$ is finite for positive fixed values of $y$.

$$\therefore\ \rho^{\alpha,\beta}_{a,b,k}\ (e^{-st}\, xF_\nu(xy)) < \infty$$

$$\therefore\ e^{-st}\, xF_\nu(xy) \in \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\ (\Omega).$$

*Lemma* 2.4—Let $|\nu| \leqslant \alpha \leqslant \frac{1}{2}$, then for fixed $t > 0$.

$$xQ(x,r,s,t) \in \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\ (\Omega),$$

where

$$Q(x,r,s,t) = \frac{2\sin p\pi \sin(p+\nu)\pi}{\pi \sin \nu\pi}\, q(x,r,s,t),\ r = x$$

and

$$q(x, r, s, t) = \frac{e^{-st}\, x^{2\nu} - r^{2\nu}}{(xr)^{\nu}(x^2 - r^2)} = e^{-st}\, q(x, r)\; r \neq x$$
$$= e^{-st}\, \frac{\nu}{x^2} = e^{-st}\, q(x, r)\; r = x. \qquad \ldots(1)$$

where

$$q(x, r) = \frac{x^{2\nu} - r^{2\nu}}{(xr)^{\nu}(x^2 - r^2)},\; r \neq x$$
$$= \frac{\nu}{x^2},\; r = x$$

[Pathak and Pandey[5], p. 250].

PROOF : The proof of the lemma is clear and parallel to Lemma 2.5 of Ahirrao and More[1].

## 3. PROPERTIES OF THE SPACE $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$

(I) $D(\Omega) \subset \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$, and the topology of $D(\Omega)$ is stronger than the induced topology on it by $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$. The restriction of $f \in \left(\mathscr{L} H^{\nu,p}_{\alpha,\beta,b,b}(\Omega)\right)$ to $D(\Omega)$ is in $D^1(\Omega)$

(II) For each $a, b, \nu, p$ $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega) \subset \epsilon(\Omega)$.

(III) For each $f \in \left(\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\right)$, $(\Omega)$ there exists a non-negative integer $r$ and a positive constant $C$ such that

$$|< f, \phi >| \leqslant C\, T_{a,b,K}(\phi)$$

where

$$T_{a,b,K}(\phi) = \max_{\substack{0 \leqslant k_1 \leqslant r \\ 0 \leqslant k_2 \leqslant r}} \rho^{\alpha,\beta}_{a,b,k}(\phi).$$

(IV) Let $f(t, x)$ be a locally integrable function such that

$\int_{\infty}^{\infty} \int_{0}^{\infty} |[\eta_{a,b,j}(t, x)]^{-1}\, x f(t, x)|\, dt\, dx$ exists, then $f(t, x)$ generates a regular generalized function on $\left(\mathscr{L} H^{\nu,p}_{\alpha,\alpha,a,b}\right)$, $(\Omega)$ through the definition

$$< f, \phi > = \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t, x)\, \phi(t, x)\, dt\, dx,\; \phi \in \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\; (\Omega),$$

4. THE CONVENTIONAL LAPLACE-HARDY $LF_\nu$ TRANSFORM ON $\mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$.

Let $\phi(t, x)$ be the conventional function defined on $\Omega = \{(t, x) \mid -\infty < t < \infty, 0 < x < \infty\}$. If $\phi(t, x) \in \mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$ then its conventional Laplace-Hardy $LF_\nu$ transform on $\Omega$ is given by

$$F(s, y) = LF_\nu(\phi(t, x)) = \int_{-\infty}^{\infty} \int_0^{\infty} e^{-st} F_\nu(xy)\, x\, \phi(t, x)\, dx\, dt \qquad \ldots(4.1)$$

exists for $a + \text{Re}\, s < 0$, $b + \text{Re}\, s < 0$, $|\nu||\nu| \leqslant \alpha \leqslant \frac{1}{2}, \leqslant 1/2$, $\alpha + \nu + 2p \geqslant 0$ and $\beta \geqslant \max(\nu + 2p - 2, -1/2)$.

Indeeed, we know that for an appropriate $M \geqslant 0$

$|F_\nu(xy)| \leqslant M(xy)^{\nu+2p}$ for all $x > 0, y > 0$ (Pathak and Pandey[6], p. 696).

$$\therefore |\bar{F}(s, y)| = \left| \int_{-\infty}^{\infty} \int_0^{\infty} e^{-st} F_\nu(xy)\, x\, \phi(t, x)\, dt\, dx \right|$$

$$\leqslant M\, y^{\nu+2p}\, \rho^{\alpha,\beta}_{a,b,0}\, (\phi(t, x)) \left\{ \int_0^{\infty} \int_0^{1} |x^{\nu+2p+2-2j-\alpha} e^{-(a+s)t}\, dt\, dx| \right.$$

$$\left. + \int_{-\infty}^{0} \int_0^{\infty} |x^{\nu+2p+\beta+4} e^{-(b+s)t}\, dt\, dx| \right\}.$$

Now consider first integral

$$\int_0^{\infty} \int_0^{1} |x^{\nu+2p-2j-\alpha+2} e^{-(a+\text{Re}s)t}\, dt\, dx|$$

$$= \int_0^{\infty 0} |e^{-(a+\text{Re}s)t}\, dt| \int_0^{1} |x^{\nu+2p-2j-\alpha+2}\, dx|$$

which exists for $a + \text{Re}\, s > 0$ $|\nu| \leqslant \alpha \leqslant 1/2$ and $\nu + 2p - \alpha - 2j + 3 \geqslant 0$. For second integral consider

$$\int_{-\infty}^{0} \int_1^{\infty} |x^{\nu+2p+\beta+4} e^{-(b+\text{Re}s)s}|\, dt\, dx$$

exists for $b + \text{Re}\, s < 0$ and $\beta < -(\nu + 2p + s)$

$\therefore$ (4.1) exists for $a + \text{Re}\, s > 0$, $b + \text{Re}\, s < 0$, $\alpha + \nu + 2p \geqslant 0$,

$\beta \geqslant \max(\nu + 2p - 2, -\frac{1}{2})$ and $|\nu| \leqslant \alpha \leqslant \frac{1}{2}$.

The inversion function $\phi(t, x)$ is represented by

$$\phi(t, x) = \frac{1}{2\pi i} \lim_{\eta \to \infty} \int_0^\infty \int_{\sigma - i\eta}^{\sigma + i\eta} F(s, y)\, e^{st}\, C_\nu(xy)\, y dy\, ds \qquad \ldots(4.2)$$

for all $(t, x) \in \Omega$ where $\eta$ be a real variable and $\sigma = \mathrm{Re}\, s$ is any fixed real number such that $\sigma_1 < \sigma < \sigma_2$.

## 5. Laplace-Hardy $\mathscr{L}F_\nu$ Transformation of Generalized Function

For $f(t, x) \in \left(\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\right), (\Omega)$ we define its distributional Laplace-Hardy $\mathscr{L}\ F_\nu$ transformation by the relation

$$F(s, y) = \mathscr{L}\ F_\nu \{f(t, x)\} = \langle f(t, x)\, e^{-st}\, x\, F_\nu(xy) \rangle \qquad \ldots(5.1)$$

where $y > 0$, $x > 0$, $t > 0$, $-\frac{1}{2} \leqslant \nu \leqslant 1/2$ and $F_\nu(xy)$ is the same as defined in section § 1. We known that for fixed $y > 0$, $e^{-st}\, xF_\nu(xy)$ belongs to $\mathscr{L}\ H^{\nu,p}_{\alpha,\beta,a,b}(\Omega)$ [Lemma 2.3] as a function of $t$ and $x$ and $f \in \left(\mathscr{L}\ H^{\nu,p}_{\alpha,\beta,a,b}\right), (\Omega)$. Therefore the relation (5.1) is meaningful.

We shall now state the analyticity theorems for the generalized Laplace-Hardy $\mathscr{L}\ F_\nu$ transformation.

*Theorem* 5.1 (The Analyticity Theorem)—For $y > 0$, let $F(s, y)$ be defined by (5.1), the $F(s, y)$ is infinitely differentiable and

$$F^k(s, y) = \langle f(t, x), \frac{\partial^{k_1}\, \partial^{k_2}}{\partial s^{k_1}\, \partial y^{k_2}}\, e^{st}\, x\, F_\nu(xy). \qquad \ldots(5.2)$$

If

$$D^k = \frac{\partial^{k_1}\, \partial^{k_2}}{\partial s^{k_1}\, \partial y^{k_2}}, \text{ then}$$

$$F^k(s, y) = \langle f(t, x)\, D^k\, e^{-st}\, xF_\nu(xy) \rangle.$$

*Theorem* 5.2 (Order property)—Let $f$ be a member of $\left(\mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b}\right), (\Omega)$, $y > 0$, $a < \mathrm{Re}\, s < b$ and $F(s, y)$ be defined by

$$F(s, y) = \langle f(t, x),\ xe^{-st}\, F_\nu(xy) \rangle,$$

then

$$F(s, y) = 0\,(\,|\,sy))^{\nu+2p} \text{ as } y \to 0,\ a < \mathrm{Re}\, s < b$$

$$= 0\,(\,|\,sy\,|\,)^{4r+\nu+p} \text{ as } y \to \infty,\ a < \mathrm{Re}\, s < b \qquad \ldots(5.3)$$

where $r$ is some non-negative integer and $\alpha$ is a fixed positive number such that $|\nu| \leqslant \alpha \leqslant 1/2$ and $\beta \geqslant 6 = \max(\nu + 2p - 2, -\frac{1}{2})$.

*Theorem* 5.3 (Boundedness Theorem)—If $F(s, y) = \mathscr{L}\ F\nu\ \{f(t, x)\}$ for $(s, y) \in \Omega' f$, then $F(s, y)$ is bounded on any subset $\Omega' f = \{(s,y) \mid a < \operatorname{Re} s < b,\ 0 < y < \infty\}$ of $\Omega$ according to

$|F(s, y)| \leqslant P(|sy|)$ where $P(|sy|)$ is a polynomial depending upon $a$ and $b$ and

$$|\nu| \leqslant \alpha \leqslant -\tfrac{1}{2}. \qquad \ldots(5.4)$$

## 6. Inversion and Uniqueness

In this section we have extended the inversion formula (4.2) to a space of generalized functions in the sense of weak distributional convergence.

The proof of inversion formula requires some lemmas and wherever necessary the proofs of some lemmas are given.

*Lemma* 6.1—Let $\mathscr{L}\ F_\nu\{f(t, x)\} = F(s, y)$, for $(t, x) \in \Omega$ and for any $\phi(t, x) \in (\Omega)$, set

$$G(s, y) = \int_0^\infty \int_{-\infty}^\infty e^{st}\, \phi(t, x)\, C_\nu(xy)\ x\,dx\,dt. \qquad \ldots(6.1)$$

Then for any pair of fixed real numbers $n$ and $n'$ with $-\infty < n < \infty$; $0 < n' < \infty$

$$(2\pi)^{-1} \int_0^{n'} y\,dy \int_{-n}^{n} G(s, y) < f(T, t),\ t\ e^{\,sT}\ F_\nu(ty) > d\omega$$

$$= < f(T, t),\ (2\pi)^{-1}\ t \int_0^{n'} \int_{-n}^{n} G(s, y)\ e^{-sT}\ y\ F_\nu(ty)\ d\omega\ dy > \quad \ldots(6.2)$$

where $s = \sigma + iw$ and $\sigma$ is any fixed real number such that $\sigma_1 < \sigma < \sigma_2$. $\operatorname{Re} s = \sigma$.

Proof : For $\phi(t, x) = 0$ on $\Omega$, the proof is trivial. Let $\phi(t, x) \neq 0$ on $\Omega$. Since $F(s, y)$ is analytic on $\Omega$ and $G(s, y)$ is entire, the integral on the right-hand side of (6.1) exists. First we shall prove that

$$V(T, t) = (2\pi)^{-1}\ t \int_0^{n'} \int_{n}^{n} G(s, y)\ e^{-sT}\ yF_\nu(ty)\ d\,w\,dy \qquad \ldots(6.3)$$

as a function of $(T, t)$ belongs to $\mathscr{L}\ H^{\nu,p}_{\alpha\cdot\beta,a,b}\ (\Omega)$.

For consider,

$$\rho^{\alpha,\beta}_{a,b,k}(V(T, t)) = \sup_{\substack{-\infty < T < \infty \\ 0 < t < \infty}} \left| \eta_{a,b,j}(T, t)\ D_T^{k_1}\ \beta_{\nu,t}^{K_2} \left( \frac{V(T, t))}{t} \right) \right|$$

(equation continued on p. 998)

$$= \sup_{\Omega} \mid \eta_{a,b,j}\ (T, t)\ D_T^{k1} \beta_{v,t}^{K2} \left\{(2\pi)^{-1} \int_0^{n'} y dy \int_{-n}^{n} G(s, y) e^{-sT} F_v(ty) dw\right\} \mid .$$

By smoothness of the integral we may carry the operator $D_T^{k1}\ \beta_{v,t}^{k2}$ under the integral sign. Hence we get

$$\rho_{a,b,k}^{\alpha,\beta}(v(T, t) = \sup_{\Omega} \mid \eta_{,ab,j}\ (T, t)\ (2\pi)^{-1} \int_0^{n'} \int_{-n}^{n} yG(s, y)\ (-1)^{k_1}\ s^{k_1}\ e^{-st}$$

$$\{(-1)^{k_2}\ y^{2k_2}\ F_v(ty) - y^{v+2p} \sum_{i=1}^{k2} a_i\ t^{v+2p-2i}\ y^{2k_2-2i}\}\ dw\ dy \mid$$

$$\leqslant \sup_{\Omega} \mid \eta_{a,b,j}\ (T, t)\ \ e^{-sT}\ t^{v+2p}\ s^{K_1}\ (2\pi)^{-1} \int_0^{n'} \int_{-n}^{n} G(s, y)\, y^{v+2p+1}$$

$$\{(ty)^{-(v+2p)}\ F_v(ty) + y^{2k_2} \sum_{i=1}^{k_2} a_i\ (ty)^{2i}\}\ dy\ dw \mid .$$

Now using the asymptotic orders of $F_v(ty)$, $\mid (ty)^{-(v+2p)}\ F_v(ty) \mid$ is bounded by $M$ for all $t > 0$, fixed $y > 0$, $-\frac{1}{2} \leqslant v \leqslant \frac{1}{2}$ and $p$ real. The series on right side in the bracket is series of positive finite terms which are bounded by say $M_1$ for $t > 0$, fixed $y > 0$ and $a_{k_2}$ being certain constants depending on $v$ and $p$.

Therefore $\rho_{a'b'K}^{\alpha,\beta}\ V(T, t) < \infty$, hence $V(T, t) \in \mathscr{L}H_{\alpha'\alpha'a'b}^{v,p}$ $(\Omega)$. Therefore the right-hand side of (6.2) is meaningful. Now consider the Reimann-Sum

$$\circledast_{M,N}(T, t) = \sum_{i=1}^{M} \sum_{i=1}^{N} te^{-s_iT}\ F_v(ty_j)\ G(s_i, y_j)\ \frac{2nn'}{MN}. \qquad ...(6.4)$$

By applying $f(T, t)$ to (6,4) term by term, we get

$$< f(T, t), \circledast_{m,N}\ (T,\ t) = \sum_{i=1}^{M} \sum_{j=1}^{N} f(T, t), te^{-s_iT}\ F_v(ty_j)$$

$$G(s_i, y_j) \frac{2nn'}{MN} >. \qquad ...(6.5)$$

Since $< f(T, t),\ te^{s_iT}\ F_v(ty_j)\ \ G(s_i\ y_j) >$ is a continuous function on $-n \leqslant w \leqslant n$, $0 < y \leqslant n'$, the sum on the right-hand said of (6.5) tends to

$$\int_0^{n} \int_{-n}^{n} < f(T, t), t\ e^{-sT}\ F_v(ty)\ G(s, y) > y\ dy\ dw, \text{ as } M \to \infty, \text{ and } N \to \infty.$$

Since $f \in \left( \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b} \right)$, $(\Omega)$ our lemma will be proven when we show that $\Theta_{M,N}(T, t)$ converges in $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$ to

$$t \int_0^n \int_{-n}^n e^{-sT} F_\nu (ty)\, G(s, y)\, yd\, ydw, \text{ as } M \to \infty,\ N \to \infty.$$

In other words we have to prove that for each pair of fixed real number $n,n'$ $|\Theta_{M,N}(T,t) - V(T, t)|$ converges uniformly to zero on $\Omega$ as $M \to \infty$, $N \to \infty$, the proof can now be carried on as in Zemanian[9] (pp. 65-66).

*Lemma* 6.2—Let $a$, $b$, $\sigma$ and $r$ be real numbers with $a < \sigma < b$. Also let $\phi(t, x) \in D(\Omega)$, then

$$(\pi)^{-1} \int_{-\infty}^{\infty} \phi(t_1 + T,\ t)\, e^{\sigma t_1} \frac{\sin rt_1}{t_1}\, dt_1$$

converges to $\phi(t_1, T)$ in $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$ as $r \to \infty$.

PROOF : The proof is parallel to Lemma 3.5.2 [Zemanian[9], pp. 66-68].

*Lemma* 6.3—For $\phi(t, x) \in D(\Omega)$, set $G(s, y)$ as in Lemma 6.1, then

$$tM_{nn'}(T,t) = t\ (2\pi)^{-1} \int_0^{n'} ydy \int_{-n}^{n} G(s, y)\, e^{-sT} F_\nu (ty)\, dw \qquad \ldots(6.6)$$

converges to $t\,\phi(T, t)$ in $\mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}$ $(\Omega)$ as $n, n' \to \infty$.

PROOF : We shall prove the result by justifying the steps in the following manipulations. Let $(a',b') \times (a'', b'')$ be the compact support of $\phi(t, x) \in D(\Omega)$. Therefore

$$(2\pi)^{-1}\ t \int_0^{n'} \int_{-n}^{n} G(s, y)\, e^{-sT}\ yF_\nu (ty)\, dwdy$$

$$= (2\pi)^{-1}\, t \int_0^{n'} \int_{-n}^{n} \int_{a'}^{b'} \int_{-\infty}^{\infty} \phi(t,x)\, e^{st}\, C_\nu (xy)\, e^{-sT}\, yF_\nu (ty)\, xd\, xdt\, dw \ldots(6.7)$$

$$= (2\pi)^{-1}\, t \int_{-p}^{n'} \int_{-\infty}^{\infty} \int_{a'}^{b'} \left(\int_0^{n'} e^{-sT}\, C_\nu (xy)\, F_\nu (ty)\, ydy\right) \phi(t, x)\, e^{st}\, xdx\, dt\, dw \qquad \ldots(6.8)$$

$$= (2\pi)^{-1}\, t \int_{-n}^{n} \int_{-\infty}^{\infty} \int_{a'}^{b'} (H_{n'}(t, x, s, T) + Q(t, x, s, T))\, \phi(t, x)\, e^{st}\, xdx\, dt\, dw \qquad \ldots(6.9)$$

$$= (2\pi)^{-1}\, t \int_{-n}^{n} \int_{-\infty}^{\infty} e^{-sT}\ e^{st}\ \phi\,(t,\, x)\, dt\ dw \qquad ...(6.10)$$

$$= (2\pi)^{-1}\, t \int_{-\infty}^{\infty} \phi\,(t,\, x)\, \Big( \int_{-n}^{n} e^{s(t-T)}\, dw\ dt \qquad ...(6.11)$$

$$= (\pi)^{-1}\, t \int_{-\infty}^{\infty} e^{\sigma(t-T)}\ \frac{\sin\,(n\,(t\,-\,T))}{(t\,-\,N)}\ \phi\,(t,\, x)\, dt \qquad ...(6.12)$$

$$= (\pi)^{-1}\, t \int_{-\infty}^{\infty} e^{\sigma t_1}\ \frac{\sin\ nt_1}{t_1}\ \phi\,(t_1\,+\,T,\, t)\, dt_1. \qquad ...(6.13)$$

This converges to

$t\,\phi\,(T,\, t)$ as $n \to \infty$. ...(6.14)

The step (6.7) follows from the expression of $G\,(s,\, y)$.

The result (6.8) follows from (6.7) by interchanging the order of integration. Since $\phi$ $(t,\, x)$ is of bounded support and the integrand in (6.7) is a smooth function of $(t,\, T)$.. Result (6.9) follows from (6.8) by Lemma 1 (Ahirrao and More[1]). The expression (6.10) follows from (6.9) by Lemma 2. (Ahirrao and More[1]). Result (6.11) follows from (6.10) by using law of indices and by interchanging the order of integration, since the integrand is a smooth function having a bounded support. Result (6.12) follows from (6.11) by integrating $\int_{-n}^{n} e^{s(t-T)}\, dw$. Result (6.13) follows from (6.12) by substituting $t = t_1 + T$ and replacing $x = t$. Result (6.14) is an immediate consequence of Lemma 6.2. This completes the proof of Lemma 6.3.

We are now ready to state and prove the inversion theorem.

*Theorem* 6.1. (Inversion Theorem)—If $f\,(t,\, x) \in \left( \mathscr{L}H^{\nu,p}_{\alpha,\beta,a,b} \right)(\Omega)$

and

$$F\,(s,\, y) = \mathscr{L}\ F_\nu\ \{f\,(t,\, x)\} = \langle f\,(t,\, x),\ e^{-st}\ x\ F_\nu\,(xy) \rangle.$$

Then in the sense of convergence in $D'\,(\Omega)$

$$f\,(t,\, x) = \lim_{n,n'\to\infty} \frac{1}{2\pi i} \int_{0}^{n'} \int_{\sigma-in}^{\sigma+in} F\,(s,\, y)\, e^{zt}\, C_\nu\,(x,\, y)\, ydy\ ds \qquad ...(6.15)$$

where $\sigma$ is any fixed real number such that $\sigma_1 < \sigma < \sigma_2$, $\mathrm{Re}\ s = \sigma$.

PROOF : Let $\phi(t, x) \in D(\Omega)$, our object is to show that

$$\lim_{n,n' \to \infty} <(2\pi i)^{-1} \int_0^{n'} \int_{\sigma-in}^{\sigma+in} F(s, y)\, e^{st}\, C_\nu(xy)\, y dy\, ds,\; x\,\phi(t, x)>$$

$$= < f(T, t),\; t\,\phi(T, t)>. \qquad ...(6.16)$$

Since $\phi(t, x) \in D(\Omega)$, $x\,\phi(t, x) \in D(\Omega)$, (6.16) will be equivalent to showing that

$$\lim_{n,n' \to \infty} <(2\pi i)^{-1} \int_0^{n'} \int_{\sigma-iw}^{\sigma+in} F(s, y)\, s^{st}\, C_\nu(xy)\, y dy\, ds,\; x\,\phi(t, x)>$$

$$= < f(T, t),\; t\,\phi(T, t)>. \qquad ...(6.17)$$

From the analyticity of $F(s, y)$ on $\Omega$, the integral (6.16) is a continuous function of $(t, x)$ and therefore the left hand side without the limit notation can be written as

$$(2\pi)^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} \phi(t, x)\, x dx\, dt \int_0^{n'} \int_{\sigma in}^{\sigma+in} F(s, y)\, e^{st}\, C_\nu(xy)\, y dy\, dw \qquad ...(6.18)$$

where $s = \sigma + iw$, since $\phi(t, x)$ is of compact support and the integrand is a continuous function of $(x, y, w, t)$, the order of integration may be changed to yield

$$(2\pi)^{-1} \int_0^{n'} \int_{-n}^{n} < f(T, t)\, e^{-sT}\, tF_\nu(ty)\, G(s,y) > y dy\, dw \qquad ...(6.19)$$

which is by lemma 6.1 equal to

$$< f(T, t), (2\pi)^{-1}\, t \int_0^{n'} \int_{-n}^{n} e^{-sT}\, F_\nu(ty)\, G(s, y)\, y dy\, dw \qquad ...(6.20)$$

because $f$ belongs to $\left( \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b} \right)$, $(\Omega)$ and in view of lemma 6.3, the last expression tends to $< f(T, t),\; t\,\phi(T, t)>$ as $n, n' \to \infty$, which completes the proof of the inversion theorem.

*Theorem* 6.2 (Uniqueness Theorem)—Let $F(s, y) = \mathscr{L} F_\nu \{ f(t, x)\}$ and $G(s, y) = \mathscr{L}\; F_\nu \{g(t, x)\}$ for all

$$(s, y) \in \Omega_f = \{(s, y) \mid \sigma_1 < \text{Re}\, s < \infty,\; 0 < y < \infty\}$$

and

$$(s, y) \in \Omega_g = \{(s, y)\; \sigma_1 < \text{Re}\, s < \infty,\; 0 < y < \infty.$$

Assume that $\Omega_f \cap \Omega_g$ is not empty and that $F(s, y) = G(s, y)$ for $(s, y) \in \Omega_f \cap \Omega_g$. Then in the sense of equality in $D'(\Omega)$ $f(t, x) = g(t, x)$ for all $(t, x) \in \Omega$,

PROOF : For any $\phi(t, x) \in D(\Omega)$ and using Theorem (6.1) we get

$$< f(t, x) - g(t, x)\, t\phi(t, x) >$$

$$= \lim_{n.n' \to \infty} < \frac{1}{2\pi i} \int_{-n}^{n} \int_{0}^{n'} e^{st}\, C_\nu(xy)\, [F(s, y) - G(s, y)]\, y\, dy\, ds,\ t$$

$$(t,x)> = 0 \text{ for any } \phi(t, x \in)\, D(\Omega).$$

Hence $f(t, x) = g(t, x)$ in the sense of equality in $D'(\Omega)$.

## 7. A Representation Theorem for the Distributional Laplace-Hardy $\mathscr{L} F_\nu$ Transformation

*Theorem* 7.1 (A Representation Theorem)—If $f \in \left( \mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b} \right)$, $(\Omega)$ then $f$ can be represented as

$$< f, \phi > \leqslant \sum_{m=0}^{k1} \sum_{n=0}^{k2} \beta_{m,n} \frac{\partial^{m+n}}{\partial t^m\, \partial x^n} \quad [\eta_{a,b,j}(t,x) P_{m,n}(x)\, F_{m,n}(t,x)]\phi > \qquad \ldots(7.1)$$

where $F_{mn}(t, x)$ are continuous functions on $\Omega$ and $P_{mn}(x)$ are polynomials of degree $k_1 + k_2$ and

$\beta_{m,n}$ = constant = max $((2j + \alpha)(2j + \alpha - 1)(2j + \alpha - 2)\ldots(2j + \alpha - k_1 + 1).$ $a^{k_1})$ $j = 0, 1, 2\ldots$

## 8. Operational Transform Formulae

The operation $\phi \to -\dfrac{\partial \phi}{\partial t}$ is a continuous linear of mapping of $\mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a\cdot b}$ $(\Omega)$ into itself. Since

$$\rho^{\alpha,\beta}_{a,b,k_1,k_2}\left(\frac{-\partial \phi}{\partial t}\right) = \operatorname*{Sup}_{\Omega} \mid \eta_{a,b,j}(t, x)\, D_t^{k_1}\, \beta_{\nu,x}^{k_2} \left(\frac{-\dfrac{\partial \phi}{\partial t}}{x}\right) \mid$$

$$= \operatorname*{Sup}_{\Omega} \mid \eta_{a,b,j}(t, x)\, D_t^{k_1+1}\, \beta_{\nu,x}^{k_2} \left(\frac{\phi(t, x)}{x}\right) \mid$$

$$= \rho^{\alpha,\beta}_{a,b,k_1+1,k_2}(\phi) \text{ for all } \phi \in \mathscr{L} H^{\nu,p}_{\alpha,\beta,a,b}\ (\Omega).$$

Therefore we define the operator $D_t$ on $\left( \mathscr{L} H^{\nu,p}_{\alpha\cdot\beta\cdot a,b} \right)$, $(\Omega)$ by the relation

$$< D_t f, \phi > = < f, -\frac{\partial \phi}{\partial t} >. \qquad \ldots(8.1)$$

This leads to the following transform formula. If $f$ is Laplace-Hardy transformable generalized function, then

$$\left(\mathscr{L} F_\nu \ \frac{\partial f}{\partial t}\right)(t, x) = \langle D_t f,\ e^{-st}\, xF_\nu(xy)\rangle$$

$$= \langle f, (-D_t)\ e^{-st}\, xF_\nu\ (xy)\rangle$$

$$= \langle f,\ s\, e^{-st}\, xF_\nu(xy)\rangle$$

$$= s\ \langle f,\ e^{-st}\, xF_\nu(xy)\rangle$$

$$\left(\mathscr{L} F_\nu \ \frac{\partial f}{\partial t}\right)(t, x) = s\ \mathscr{L}\ F_\nu\, \{f(t, x)\}, \text{ for all } (t, x) \in \Omega.$$

More generally, for $K = 1, 2, 3 \ldots$

$$\left(\mathscr{L} F_\nu\ D_t^k\, f\right)(t, x) = s^k\ \mathscr{L}\ F_\nu\, \{f(t, x)\},\ (t, x) \in \Omega. \qquad \ldots(8.2)$$

Similarly, the operation $\phi \to \Delta_x$ is a continuous linear mapping of $\mathscr{L}\, H^{\nu, p}_{\alpha, \beta, a, b}$ $(\Omega)$ into itself since

$$\rho^{\alpha, \beta}_{a, b, k_1, k_2}\quad (\Delta_x\, \phi) = \operatorname*{Sup}_{\Omega}\ |\ \eta_{a,b,j}\ (t, x)\ D_t^{k_1}\ \ \beta^{k_2}_{\nu, x}\ \left(\frac{\Delta_x \phi}{x}\right)|$$

$$= \rho^{\alpha, \beta}_{a, b k_1, k_2 + 1}\ \ (\phi).$$

We define the operator $\Delta_x$ on $\left(\mathscr{L}\ H^{\nu, p}_{\alpha, \beta, a, b}\right)$, $(\Omega)$ by $\langle \Delta_x f, \phi\rangle = \langle f, -\Delta_x\, \phi\rangle$.

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# DISTRIBUTIONAL BOUNDARY VALUES IN $\left(W_M^\Omega\right)'$-SPACES OF FUNCTIONS HOLOMORPHIC IN TUBE DOMAINS

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Fourier-Laplace transforms of generalized functions in $\left(W_M^\Omega\right)'$ -spaces are studied. The results are used to obtain distributional boundary values of functions which are holomorphic in tube domains $T^C = \mathbb{R}^n + i\,C$ and which satisfy certain growth conditions. Converse results are also obtained.

## 1. INTRODUCTION

Solov'ev[11] has extended Schwartz's theory of functionals of arbitrary singularity and studied the Fourier-Laplace transform of $S_\alpha^\beta$ -spaces as distributional boundary values of functions holomorhhic in tube domains. Many other authors have studied distributional boundary values concerning functions holomorphic in tube domains in the papers[1 4,7-9].

Gelfand and Shilov[6] have explored the spaces $W_M^\Omega$ and $\left(W_M^\Omega\right)'$. In this paper we establish the conditions for a generalized function in $\left(W_M^\Omega\right)'$ to have a Fourier-Laplace transform and describe growth conditions of spectral functions. Imposing certain growth conditions on holomorphic functions in tube domains, distributional boundary values in $\left(W_M^\Omega\right)'$-spaces are studied. Converse results are obtained by assuming certain growth conditions on spectral functions.

## 2. NOTATION AND DEFINITIONS

The notation and terminology of this work will follow that of Schwartz[10]. For various $n$ dimensional notations and definitions of various terms related to tubular radial domains we refer to Pathak[8,9] and Schwartz[12].

Suppose that $\Phi\ (\mathbb{R}^n)$ and $\Psi\ (\mathbb{R}^n)$ are test function spaces and $\Phi'$, $\Psi'$ are their respective dual spaces. The Fourier transform of the function $\phi \in \Phi$ is an element of $\Psi$ defined by

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$$\mathscr{F}[\phi(t)] = \tilde{\phi}(t) = \Psi(\sigma) = \int_{\mathbb{R}^n} \phi(t)\, e^{i<\sigma\cdot t>}\, dt.$$

$t \in \mathbb{R}^n$; while the inverse Fourier transform is defined by

$$\mathscr{F}^{-1}[\phi(t)] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi(t)\, e^{i<\sigma\cdot t>}\, dt.$$

Suppose the Fourier transform defined above maps $\Phi$ onto $\Psi$ one-to-one and continuously. Then the Fourier transform $\Psi'$ into $\Phi'$ is defined by

$$< \mathscr{F}[g], \phi > = < g, \mathscr{F}[\phi] >.$$

If on $\Psi$, the operation of multiplication by $e^{-<\sigma\cdot y>}$, $y \in \mathbb{R}^n$, is defined and continuous, then for each generalized function $g \in \Psi'$ $\mathscr{F}\,]g\, e^{<\sigma\cdot y>}] \in \Phi$.

Let $f(z)$ be a holomorphic function in $T^C = \mathbb{R}^n + iC$, where $C$ is an open connected cone. A generalized function $g$ is said to be the spectral function of $f(z)$ if

$$< \mathscr{F}[g(\sigma)\, e^{-<\sigma\cdot y>}], \phi > = \int_{\mathbb{R}^n} f(x + iy)\, \phi(x)\, dx \qquad \ldots(2.1)$$

for all $y \in C$, $\phi \in \Phi$. The holomorphic function $f(z)$ is called the Fourier-Laplace transform of $g$. The function $(g^* \Psi)(\tau)$ defined by

$$(g * \Psi)(\tau) = < g(\sigma), \Psi(\sigma + \tau) > \qquad \ldots(2.2)$$

where $g \in \Phi'$ and $\Psi \in \Phi$, plays a fundamental role in our present investigation.

## 3. Test Function Spaces and Their Duals

Let $M(x)$, $(x \geqslant 0)$, and $\Omega(y)$, $(y \geqslant 0)$ be arbitrary convex functions defined by Gelfand and Shilov[6] (pp. 1-6). Let $W_M^{a} = W_{M_1, \ldots M_n}^{a_1, \ldots a_n}$ be the test function space defined by Gelfand and Shilov[6] (p.26). Then the following results hold :

$$M(x_1) + M(x_2) \leqslant M(x_1 + x_2)\; x_1, x_2 \geqslant 0.$$

Assume that for sufficiently large $x$

$$M_1(\gamma_1, x) \leqslant M_2(\gamma_2\, x)$$

with some constants $\gamma_1$ and $\gamma_2$. Then the inclusion

$$W_{M_1}^{a} \supset W_{M_2}^{a}$$

holds. If $\Omega_1(y) \leqslant \Omega_2(y)$ for $y \geqslant 0$, then

$$W_M^{a_1} \subset W_M^{a_2}.$$

The dual space of $W_M^{\Omega}$ is denoted by $\left(W_M^{\Omega}\right)'$. If $M(x)$ nnd $\Omega(y)$ are dual in the sense of Young then

$$xy \leqslant M(x) + \Omega(y) \text{ for } x, y \geqslant 0. \qquad \ldots(3.1)$$

For any $x$ we can find a $y = y(x)$ which with the given $x$ turns the inequality (3.1) into an equality.

Let us note a few properties of dual functions which will be useful in the sequel.

*Lemma* 3.1—If $M(x)$ is dual to to $\Omega_d(y)$, $M_1(x)$ is dual to $\Omega_{d_1}(y)$ and $M(x) < M_1(x)$ for sufficiently large values of $x$, then $\Omega_d(y) > \Omega_{d_1}(y)$ for sufficiently large $y$.

Proof : See Gelfand and Shilov[6] (p. 19)

Let $\Omega_d(y)$ and $M_d(x)$ be the functions which are dual in the sense of Young to the functions $M(x)$ and $\Omega(y)$, respectively, then we have the following relation[6] (p. 27):

$$\widetilde{W}_M^{\Omega} = \mathscr{F}\left[W_M^{\Omega}\right] = W_{M_d}^{\Omega_d} \qquad \ldots(3\ 2)$$

Let $\left(W_M^{\Omega}\right)'$ be the dual space of $W_M^{\Omega}$. Then the Fourier transform of an element $V \in \left(W_M^{\Omega}\right)'$ is defined to be the element $U$ such that the Parseval relation

$$<U, \psi> = (2\pi)^n < V, \phi> \qquad \ldots(3.3)$$

holds, where $\phi \in W_M^{\Omega}$, $\psi = \mathscr{F}[\phi(t)]$ belongs to $W_{M_d}^{\Omega_d}$ and $U$ is the Fourier transform of $V$ denoted by $U = \mathscr{F}[V]$. In fact the Fourier transform is an isomorphism from $\left(W_{M_d}^{\Omega_d}\right)'$ onto $\left(W_M^{\Omega}\right)'$.

## 4. Fourier-Laplace Transform on $\left(W_M^{\Omega}\right)'$-spaces

In this section we introduce the conditions for existence of a Fourier-Laplace Transform on $\left(W_M^{\Omega}\right)'$-spaces. Throughout this section, we assume that $\Omega_d$, $M_d$, $\Omega_{d_1}$ $M_{d_1}$, $\Omega_{d_0}$ and $M_{d_0}$ are the dual functions of $M$, $\Omega$, $M_1$, $\Omega_1$, $M_0$, and $\Omega_0$ respectively.

*Definition* 4.1—If $M < M_1$, then we define continuation of $g \in \left(\widetilde{W}_{M_1}^{\Omega}\right)' = \left(W_{M_d}^{\Omega_{d_1}}\right)'$ to $\hat{g} \in \left(\widetilde{W}_M^{\Omega}\right)'$ by

$$<\hat{g}, \psi> = \int_{\mathbf{R}^n} <g(\sigma), \psi_0(\sigma+\tau)\,\psi(\sigma)> d\tau \qquad \ldots(4.1)$$

where

$$\Psi \in \left(\widetilde{W}^{\Omega}_{M_1}\right) \text{and } \Psi_0 \in \left(\widetilde{W}^{\Omega}_{M_0}\right)$$

$$M < M_1 < M_0 \text{ with } \int_{\mathbf{R}} \psi_0(\sigma)\, d\sigma = 1.$$

*Lemma* 4.2—A functional $g \in \left(\widetilde{W}^{\Omega}_{M_1}\right)' = \left(W^{\Omega d_1}_{M_d}\right)'$ allows continuation to the space $\left(\widetilde{W}^{\Omega}_{M}\right) = \left(W^{\Omega d}_{M_d}\right)$, $M < M_1$ if and only if

$$|\ (\hat{g} * \psi)(\tau)\ | \leqslant \|\hat{g}\|_{ab}\, \|\psi\|_{ab}\, e^{-M_d(a\tau)}. \qquad \ldots(4.2)$$

PROOF : Let $g$ allow continuation to the functional $\hat{g} \in \left(\widetilde{W}^{\Omega}_{M}\right)'$. Then for all $a$ and $b$

$$|(\hat{g} * \psi)\ (\tau)\ | = |\ <\hat{g}(\sigma), \psi(\sigma + \tau)>\ |$$

$$\leqslant \|\hat{g}\|_{ab}\, \|\psi\,(\sigma + \tau)\|_{ab}.$$

Considering imaginary part as zero we calculate the $\|\psi\,(\sigma + \tau)\|_{ab}$ as follows :

$$\|\psi\,(\sigma + \tau)\|_{ab} = \sup_{\sigma}\ |\ \psi\,(\sigma + \tau)\ |\ e^{M_d(a\sigma)}$$

$$= \sup_{\sigma}\ |\ \psi\,(\sigma)\ |\ e^{M_d(a\sigma - a\tau)}$$

$$\leqslant \|\psi\|_{ab}\ e^{\ M_d(a\tau)}.$$

Conversely, suppose $g \in \left(\widetilde{W}^{\Omega}_{M_1}\right)'$ with the given condition (4.2). Then the continuation of $g$ is as in (4.1). Indeed the shift operation does not carry one out of $\widetilde{W}^{\Omega}_{M_0}$ and is continuous with respect to $\tau$. Now we show that a function $\psi_0 \in \widetilde{W}^{\Omega}_{M_0}$ is a multiplier in $\widetilde{W}^{\Omega}_{M_1}$ where $M_1 < M_0$. For every $\psi \in \widetilde{W}^{\Omega}_{M_1}$, we have

$$|\ \psi\,(z)\,\psi_0\,(z)\ | \leqslant L \exp\left(-M_d(ax) + \Omega_{d_1}(dy)\right) L_1 \exp\left(-M_d(ax) + \Omega_{d_0}(dy)\right)$$

$$\leqslant LL_1 \exp\left(-\ 2M_d\ (ax) + 2\Omega_{d_1}(by)\right)$$

$$\leqslant\ L'\ \exp\left(-M_d(ax) + \Omega_{d_1}(2\,by)\right) \text{ (By Lemma 3.1)}$$

Therefore, $\psi\,(z)\,\psi_0\,(z) \in \widetilde{W}^{\Omega}_{M_1}$ and hence $\psi_0 \in \widetilde{W}^{\Omega}_{M_0}$ is a multiplier in $\widetilde{W}^{\Omega}_{M_0}$. There-

fore, the integrand (4.1) is defined for any $\psi_0 \in \widetilde{W}^{\Omega}_{M_0}$ and moreover, is a continuous function of $\tau$ for which we have

$$| < g(\sigma), \psi_0(\sigma + \tau)\, \psi(\sigma) > | = | (g * \chi_\tau)(\tau) |$$
$$\leqslant \|g\|_{ab}\, \|\chi\|_{ab}\, e^{-M_d(a\tau)} \qquad \ldots(4.3)$$

where

$$\chi_\tau(\sigma) = \psi_0(\sigma)\, \psi(\sigma - \tau).$$

Now,

$$\|\chi_\tau(\sigma)\|_{ab} \leqslant |\chi_\tau(\sigma)|\, e^{M_d(a\sigma)}$$
$$= |\psi_0(\sigma)|\, \psi(\sigma - \tau)|\, e^{M_d(a\sigma)}$$
$$= \|\psi_0(\sigma)\|_{ab}\, |\psi(\sigma - \tau)|$$
$$= \|\psi_0(\sigma + \tau)\|_{ab}\, |\psi(\sigma)|.$$

Indeed

$$\|\psi_0(\sigma + \tau)\|_{ab} \leqslant |\psi_0(\sigma + \tau)|\, e^{M_d(a\sigma)}$$
$$= |\psi_0(\sigma)|\, e^{M_d(a\sigma - a\tau)}$$
$$\leqslant \|\psi_0\|_{ab}\, e^{-M(a\tau)}.$$

Therefore

$$\|\chi_\tau(\sigma)\|_{ab} \leqslant \|\psi_0\|_{ab}\, |\psi(\sigma)|\, e^{-M_d(a\tau)}. \qquad \ldots(4.4)$$

Inequalities (4.3) and (4.4) ensure the convergence of (4.1) for any $\psi_0 \in \widetilde{W}^{\Omega}_{M_0}$. By the same token the integral defines a linear functional over $\widetilde{W}^{\Omega}_{M}$. It is also continuous due to the presence of $|\psi(\sigma)|$ on the right of (4.4).

Now it remains to prove that on $\widetilde{W}^{\Omega}_{M_1}$, the functional coincides with the original that is for $\psi \in \widetilde{W}^{\Omega}_{M_1}$ integral in (4.1) can be performed within the functional bracket. For this following (4.1) we represent the space $\widetilde{W}^{\Omega}_{M_1}$ as the union of countably normed spaces

$$\widetilde{W}^{\Omega, b}_{M_1, a} = \bigcap_{\substack{a' > a \\ a. > b}} \widetilde{W}^{\Omega, b'}_{M_1, a'}.$$

Suppose $\psi \in \widetilde{W}^{\Omega}_{M_1}$. Then we have $\|\chi_\tau\|_{ab} \leqslant L\, |\psi(\sigma)|\, e^{-M_d(a\tau)}$. It follows that the

integral (4.1) in this case remains convergent if $g$ is replaced by any functional $f \in \left(\widetilde{W}^{\Omega}_{M_1}\right)'$. Indeed

$$\begin{aligned} &| < f(\sigma),\ \psi_0(\sigma+\tau)\,\psi(\sigma)> | \\ &\leqslant \| f \|_{ab}\, \| \chi_r(\sigma+\tau) \|_{ab} \\ &\leqslant \| f \|_{ab}\, | \chi_r(\sigma+\tau) |\, e^{M_d(a\sigma)} \\ &= \| f \|_{ab}\, | \chi_r(\sigma) |\, e^{M_d(a\sigma - a\tau)} \\ &\leqslant \| f \|_{ab}\, \| \chi_r(\sigma) \|_{ab}\, e^{-M_d(a\tau)}. \end{aligned}$$

Thus, the sequence of the integral sums $\Sigma\, \psi(\sigma+\tau k)\ \psi\ (\sigma)\,\Delta\,\tau k$ is weakly fundamental ia $\widetilde{W}^{\Omega}_{M_1}$ and therefore converges in $\widetilde{W}^{\Omega}_{M_1}$ to some element. This element cannot be anything else but the function $\psi$. Since convergence in the topology of $\widetilde{W}^{\Omega}_{M_1}$ implies convergence of the functions in the ordinary sense.

Since $\widetilde{W}^{\Omega}_{M_0}$ is dense in $\widetilde{W}^{\Omega}_{M_1}$ and hence in $\widetilde{W}^{\Omega}_{M}$, therefore the continuation of $g$ to $\widetilde{W}^{\Omega}_{M}$ is unique.

This proves the lemma.

*Lemma* 4.3—Let the function $f(z)$ be holomorphic in the tube domain $T^C = \mathbb{R}^n + iC$ and for all number $\epsilon > 0$ and compactum $K \subset C$, let it satisfy

$$| f(z) | \leqslant D_\epsilon(K)\, e^{\epsilon M(ax)},\ z \in \mathbb{R}^n + iK. \qquad \ldots(4.5)$$

Then $f(z)$ has a spectral function $g$ defined on some space $\widetilde{W}^{\Omega}_{M}$ that possesses the following properties :

(A) The function $g$ satisfies the growth condition $| g | \leqslant Le^{-M_d(a\tau)}$.

(B) The function $g(\sigma)\ e^{-<\sigma, y>}$ for every $y \in C$ decreases not slower then $e^{-M_d(\tau/b)}$ and this decrease is uniform with respect to $y$ in any compactum $K \subset C$.

Proof : Suppose $\phi(z) \in W^{\Omega}_{M}$. Then $\phi(z)$ is an entire analytic function which satisfies the inequality

$$| \phi(z) | \leqslant L \exp(-M(ax) + \Omega(by)). \qquad \ldots(4.6)$$

Therefore

$$<g, \psi> = \int_{\mathbb{R}^n} f(x+iy)\, \phi(x-iy)\, dx \qquad \ldots(4.7)$$

where $\phi(x) = \mathscr{F}^{-1}[\psi(\sigma)]$ and $y$ is an arbitrary point in $C$, defines a linear functional on the space $W_M^\Omega$. In view of the Cauchy-Poincare Theorem, (4.7) does not depend on on $y$. Using (4-5) and (4.6) we have

$$|<g, \psi>| \leqslant \int_{\mathbb{R}^n} |f(x+iy)| \, |\phi(x-iy)| \, dx$$

$$\leqslant D_\varepsilon(K) \int_{\mathbb{R}^n} e^{\epsilon M(ax)} \, L \, e^{-M(ax)+\Omega(by)} \, dx$$

$$= D_\varepsilon(K) \, L \, e^{\Omega(by)} \int_{\mathbb{R}^n} e^{-(1-\epsilon)M(ax)} \, dx$$

$$< \infty.$$

Therefore (4.7) is convergent. Formula (4.7) is obtained from the relation (2.1) by substituting $\phi(x) \rightarrow \phi(x-iy)$ under which the space $W_M^\Omega$ is mapped onto itself one-to-one. Therefore, the functional $g$ is the spectral function of the function $f(z)$.

Considering zero as imaginary part, we can calculate the function $g * \psi$ as follows :

$$|(g * \psi)(\tau)| = |<g(\sigma), \psi(\sigma+\tau)>|$$

$$\leqslant \|g\|_{ab} \, \|\psi(\sigma+\tau)\|_{ab}$$

$$= \|g\|_{ab} \sup_\sigma |\psi(\sigma+\tau)| \, e^{M_d(a\sigma)}$$

$$= \|g\|_{ab} \sup_\sigma |\psi(\sigma)| \, e^{M_d(a\sigma - a\tau)}$$

$$\leqslant \|g\|_{ab} \, \|\psi\|_{ab} \, e^{M_d(a\tau)}. \qquad ...(4.8)$$

Thus $g$ satisfies the condition (A).

Now suppose that $K$ is a compactum in $C$ and $Q$ is a bounded set in $\tilde{W}_M^\Omega$. Let us estimate the function $(g\, e^{-<\sigma, y>} * \psi)(t)$, $y \in K$, $\psi \in Q$. Let $\delta$ be a constant less than the distance from the compactum $K$ to the boundary of $C$. For $\|\Delta y\| \leqslant \delta$, we have

$$(g\, e^{-<\sigma, y>} * \psi)(\tau) = \int_{\mathbb{R}^n} f(x+i(y-\Delta y)) \, \phi(x - i\Delta y)$$

$$\exp(-i<\tau, x - i\Delta y>) \, dx$$

$$= \int_{\mathbb{R}^n} f(x+i(y-\Delta y)) \, \phi(x-i\Delta y) \exp(-i<\tau, x>)$$

$$\exp(-<\tau, \Delta y>) \, dx. \qquad ...(4.9)$$

Using (4.5) and (4.6) we have from (4.9) the estimate

$$| (g\, e^{-<\sigma,y>} * \psi)(\tau) |$$

$$= D_\epsilon (K)\, e^{-<\tau\cdot\Delta y>}\, e^{\Omega(b\Delta y)} \int_{\mathbb{R}^n} \exp - (1-\epsilon)\, M(ax)\, dx$$

$$= D_\epsilon' (K)\, e^{-<\tau\cdot\Delta y>}\, e^{\Omega(b\Delta y)}. \qquad \ldots(4.10)$$

For any given $\tau$ we can choose $\Delta y$ such that $< \tau, \Delta y > ==$.

$M_d(\tau/b) + \Omega(b\,\Delta\, y)$ holds. Therefore the estimate (4.10) becomes

$$| g\, e^{-<\sigma\cdot y>} * \psi)(\tau) | = D'_\epsilon\, (K)\, e^{-M_d\,(\tau/b)}$$

which satisfies the condition (B).

This proves the lemma.

*Lemma* 4.4 – Let a generalized function $g \in \left(\widetilde{W}_M^{\Omega}\right)'$ possess the property (B) with respect to some domain $C$. Then it has a Fourier-Laplace transform $f(z)$ that is holomorphic in the tube domain $T^C = \mathbb{R}^n + iC$ and satisfies in the domain the estimate

$$| f(z) | \leqslant D\,(K)\, e^{-M(ax)}. \qquad \ldots(4.11)$$

PROOF : Let us consider a function $\psi_0 \in \widetilde{W}_{M_0}^{\Omega}$, where $M_0(x) > M(x)$ with the property that $\int_{\mathbb{R}^n} \psi_0(\sigma)\, d\sigma = 1$.

Consider the expression

$$F(Z, \tau) = < g(\sigma), \psi_0(\sigma + \tau)\, e^{i<\sigma\cdot z>} >. \qquad \ldots(4.12)$$

First we show that the function $F(z, \tau)$ is holomorphic in $z$ for any $\tau \in \mathbb{R}^n$. Taking partial derivative, we have

$$\frac{\partial}{\partial z_k}(F(z, \tau)) = < g(\sigma), \psi_0(\sigma + \tau)\, i\, \sigma_k\, e^{i<\sigma\cdot z>} >.$$

Further by virtue of the condition (B) for a sufficiently small neighbourhood $U$ of an arbitrary point $z_0 \in T^C$ one can find constant $L(U)$ such that

$$| F(z, \tau) | = | < g(\sigma), \psi_0(\sigma + \tau)\, e^{i<\sigma\cdot z>} > |$$

$$= | (g\, e^{-<\sigma\cdot y>} * \psi_0 e^{i<\sigma\cdot x>})(\tau) |$$

$$\leqslant L\, e^{-M_d(\tau/b)},\ z \in U$$

and

$$\left| \frac{\partial}{\partial z_k}(F(z, \tau)) \right| \leqslant \sigma_k\, | (g(\sigma)\, e^{-<\sigma\cdot y>} * \psi_0\, e^{i<\sigma\cdot x>}) |$$

$$\leqslant \sigma_k\, L\, e^{-M_d(\tau/b)} = L'\, e^{-M_d(\tau/b)}. \qquad \ldots(4.13)$$

Therefore, $F(z, \tau)$ is continuous. It follows that the function $f(z) = \int_{\mathbb{R}^n} F(z, \tau)\, d\tau$ is defined in $T^C$ and moreover, is holomorphic in this domain.

We now estimate the behaviour of $F(z, \tau)$ not only with respect to $\tau$ but also with respect to $x$, assuming that $y$ belongs to compactum $K \subset C$. Using the continuity of the Fourier operator, we can rewrite the condition (B) as

$$| g\, e^{-<\sigma \cdot y>} * \mathscr{F}[\phi])(\tau) | \leqslant L_{ab}(K)\, \|\phi\|_{ab}\, e^{-M_d(\tau/b)} \qquad \ldots(4.14)$$

$y \in K, \phi \in W'^{\Omega}_{M}$, which on account of the relation

$$\psi_0(\sigma)\, e^{i<\sigma \cdot x>} = \mathscr{F}[\phi_0(\xi + x)]$$

ives

$$| F(z, \tau) | \leqslant L_{ab}(K)\, \|\phi_0(\xi + x)\|_{ab}\, e^{-M_d(\tau/b)}$$

$z \in \mathbb{R}^n + iK$. Now, considering imaginary part as zero, we have

$$\|\phi_0(\xi + x)\|_{ab} = \sup_{\xi} | \phi_0(\xi + x) |\, e^{M_0(a\xi)}$$

$$= \sup_{\xi} | \phi_0(\xi) | \exp(M_0(a\xi - ax))$$

$$\leqslant \|\phi_0\|_{ab}\, e^{-M_0(ax)}$$

$$\leqslant \|\phi_0\|_{ab}\, e^{-M(ax)}.$$

Hence

$$| f(z) | \leqslant \int_{\mathbb{R}^n} | F(z, \tau) |\, d\tau$$

$$\leqslant L_{ab}(K)\, \|\phi_0\|_{ab}\, e^{-M(ax)} \int_{\mathbb{R}^n} e^{-M_d(\tau_1 b)}\, d\tau$$

$$\leqslant L_{ab}(K)\, \|\phi_0\|_{ab}\, e^{-M(ax)}$$

which satisfies the estimate (4.11).

*Lemma* 4.5—Let $g \in \left(\widetilde{W}^{\Omega}_{M_1}\right)'$. If $g\, e^{-<\sigma \cdot y>}$ for every $y$ in the domain $C$ allows extension to the space $\widetilde{W}^{\Omega}_{M}$, $M < M_1$, then the functional $g$ the property (B).

Proof : Suppose $K$ is a compactum in $C$ and $Q$ is a bounded set in $\widetilde{W}^{\Omega}_{M_1}$, we represent the function $(g e^{-<\sigma \cdot y>} * \psi)$, $y \in K, \psi \in Q$ in the form

$$(g\, e^{-<\sigma \cdot y>} * \psi)(\tau)$$

$$= (g\, e^{-<\sigma \cdot y + \Delta y>} * \psi e^{<\sigma' \Delta y>})(\tau)\, e^{-<\tau' \Delta y>} \qquad \ldots(4.15)$$

assuming that $\|\Delta y\| < \delta$, and that $\delta$ is so small that $y + \Delta y$ does not leave the compactum $K \subset C$ when $y$ ranges over the compactum $K$. For the functional $\hat{g}\, e^{-<\sigma, y>}$ which continues $g\, e^{-<\sigma, y>}$ to the space $\widetilde{W}_M^{\Omega}$, we have

$$|\,(\hat{g}\, e^{-<\sigma, y>} * \chi)(\tau)\,| \leqslant \|\hat{g}\, e^{-<\sigma, y>}\|_{ab} \|\chi(\sigma + \tau)\|_{ab}$$

$$\leqslant \|\hat{g}\, e^{-<\sigma, y>}\|_{ab}\, \|\chi\|_{ab}\; e^{-M_d(a\tau)}$$

where $\chi \in \widetilde{W}_M^{\Omega}$. Therefore, using the freedom in the choice of $\Delta y$ in the representation (4.15) we obtain

$$|\,(\hat{g}\, e^{-<\sigma, y>} * (\psi)(\tau)\,| \leqslant \|\hat{g}\, e^{-<\sigma, y>}\|_{ab}\, L_{ab}(Q)\, e^{-<\tau, \Delta y> - M_d(a\tau)} \qquad \ldots(4.16)$$

where the set

$$\{\chi : \chi(\sigma) = \psi(\sigma)\, e^{<\sigma, \Delta y>},\ \psi \in Q,\ \|\Delta y\| \leqslant \delta\}$$

is bounded. The first factor on the right side of (4.16) is bounded due to the following proposition.

*Proposition* 4.6—Under the same conditions of Lemma 4.5, the function $G(y) = \|\hat{g}\, e^{-<\sigma, y>}\|_{ab}$ is continuous in the domain $C$.

We shall prove the proposition later. Now for any given $\tau$ we can choose $y$ such that Young's equality holds, that is, $<\tau, \Delta y> = M_d(a\tau) + \Omega(\Delta y/a)$. Therefore we have

$$e^{-<\tau, \Delta y> - M_d(a\tau)} = e^{-M_d(a\tau) - \Omega(\Delta y/a) - M_d(a\tau)}$$

$$= e^{-2M_d(a\tau) - \Omega(\Delta y/a)}$$

$$\leqslant e^{\;M_d(a\tau)}$$

where $\Omega(\Delta y/a) < \Omega(\delta/a)$ a constant and $e^{-\Omega(\delta/a)} < 1$. Therefore, we can conclude that for the function $(\hat{g}\; e^{-<\sigma, y>} * \psi)$, $y \in K$, $\psi \in Q$, the condition (B) holds in which $\delta$ has its earlier meaning of the distance from the compactum $K$ to the boundary of the domain $C$.

*Proof of the Proposition* 4.6—Let $y \in C$. On account of the triangle inequality for all increments $\Delta y$ that does not take one out of $C$

$$|\,G(y + \Delta y) - G(y)\,| \leqslant \|\hat{g}\, e^{-<\sigma, y + \Delta y>} - \hat{g}\, e^{-<\sigma, y>}\|_{ab}$$

$$= \sup_{\|\psi\|_{ab} \leqslant 1} |<\hat{g}\, e^{-<\sigma, y>}, (e^{-<\sigma, \Delta y>} - 1)\,\psi>|\,.$$

Strictly speaking, this equality holds only if the operation of multiplication by $(e^{-<\sigma\cdot\Delta y>}-1)$ does not carry the set $\|\psi\|_{ab} \leqslant 1$ out of $\widetilde{W}^{\Omega}_{M}$; however, for sufficiently small $\Delta\, y$ this is certainly so[11]. Moreover,a simple calculation by means of Leibnitz's rule shows that for $\|\Delta y\| \leqslant \min\left[1, \frac{1}{a} - \frac{1}{b+1}\right]$ the image of this sphere is bounded in the norm $\|\cdot\|_{a+1,b+1}$ by $L\,\|\Delta y\|$, where $L$ is a constant. Therefore, our estimate can be continued as follows

$$| G(y+\Delta y) - G(y) | \leqslant \|g\, e^{-<\sigma\cdot y>}\|_{a+1,b+1}$$

$$\times \sup_{\|\psi\|_{a,b} \leqslant 1} \|(e^{-<\sigma\cdot\Delta y>}-1)\,\psi\|_{a.1,b+1}$$

$$\to 0 \text{ as } \Delta\, y \to 0.$$

This proves the proposition.

*Lemma* 4.7— Let $g \in \left(\widetilde{W}^{\Omega}_{M_1}\right)'$. The set of all $y$ for which $g\, e^{-<\sigma\cdot y>}$ allows extension to the space $\widetilde{W}^{\Omega}_{M}$ with $M < M_1$ is convex.

PROOF : By Lemma 4.2 it is sufficient to prove that if the functionals $g\, e^{-<\sigma\cdot y_1>}$ and $g\, e^{-<\sigma\cdot y_2>}$ have growth order not higher than the first, then $g\, e^{-<\sigma\cdot y>}$, where $y = t_1\, y_1 + t_2\, y_2,\ t_1 \geqslant 0,\ t_2 \quad 0,\ t_1 + t_2 = 1$ has the same behaviour at infinity. For this we consider the estimate

$$(g\, e^{-<\sigma\cdot y>} * \psi)(\tau) = (g\, e^{-<\sigma\cdot y+\Delta y>} * \psi\, e^{<\sigma\cdot\Delta y>})(\tau)\, e^{-<\tau\cdot\Delta y>}$$

taking $\Delta\, y \parallel y_p - y,\ p = 1,2$, so that

$$(g\, e^{-<\sigma\cdot y>} * \psi)(\tau) = (g\, e^{<\sigma\cdot y_p} * \psi\, e^{<\sigma\cdot y_p - y})(\tau) e^{-<\tau\cdot y_p\ y>}.$$

Let $\psi$ range over a bounded set $Q \subset \widetilde{W}^{\Omega}_{M_0}$ with $M < M_0$. Then

$$| \psi(\sigma)\, e^{<\sigma,(y_p - y)>} | = | \psi(\sigma) | \; | e^{<\sigma\cdot(y_p\ y)>} |$$

$$\leqslant e^{-M\, d(a\sigma)}\, | e^{<\sigma\cdot(y_p - y)>} | .$$

Now, for $p = 1,2$, we have

$$\Sigma\, t_p <\sigma, y_p - y> = \sigma, (y_1 - y)\, t_1 + t_2\,(y_2 - y)> = <\sigma, 0> = 0$$

and the scalar product $<\sigma, y_p - y>$ cannot be simultaneously positive. Therefore $e^{<\sigma, y_p - y>} \leqslant 1$. So that $e^{<\sigma\cdot y_p - y>}\, \psi(\sigma)$ is bounded in $\widetilde{W}^{\Omega}_{M_0}$. That is the image of the set $Q$ under multiplication by $e^{<\sigma\cdot y_p - y>}$ is bounded in $\widetilde{W}^{\Omega}_{M_0}$. Let this image be denoted by $Q_k$. Taking minimum with respect to $p = 1,2$, we obtain

$$| (g\, e^{-<\sigma\cdot y>} * \psi)(\tau) | \leqslant \max\,[L_{k,\varepsilon}(Q)]\, e^{-M\, d(a\tau)} \min e^{-<\tau\cdot_p y - y>}.$$

Note that $e^{<r \cdot y_p \cdot y>} < 1$. This proves the lemma.

Let us summarize that we have obtained.

*Theorem* 4.8—Let a generalized function $g$ be defined on the space $\widetilde{W}^{\Omega}_{M_1}$. If $ge^{-<\sigma \cdot y>}$ for every $y$ in some domain $C$ allows extension to the space $\widetilde{W}^{\Omega}_{M}$ with $M < M_1$, then $g$ has a Fourier-Laplace transform $f(z)$ that is holomorphic in the tube domain $T^{o(C)} = \mathbb{R}^n + i\,O(C)$, where $O(C)$ is the convex hull of $C$ and for any compactum $K \subset O(C)$. The transform satisfies the estimate (4.5).

Conversely, any function that is holomorphic in the tube domain $T^C = \mathbb{R}^n + iC$ and satisfies the estimate (4.5) then is the Fourier-Laplace transform of a uniquely defined generalized function $g \in \left(\widetilde{W}^{\Omega}_{M}\right)'$ for every compactum $K \subset O(C)$.

## 5. Distributional Boundary Values of $\left(\widetilde{W}^{\Omega}_{M}\right)'$ —space

*Definition* 5.1—Let $f(x + iy)$ be holomorphic in $T^C$ and let it satisfy estimate (4.5). If a point $y$ within the domain $C$ approaches any point $\bar{y}$ of its boundary, then $f(x + iy) = g\,e^{-<\sigma \cdot y>} \rightarrow g\,e^{-<\sigma \cdot \bar{y}>}$. And the limiting value on the real space Im $z = 0$ (if $0 \in \partial C$) is the Fourier transform of the spectral function $g$.

If $C'$ is a compact subcone of $C$ we write $C' \subset\subset C$.

*Theorem* 5.2—Let the function $f(z)$ be holomorphic in the tube domain $T^{C_r} = \mathbb{R}^n + i\,C_r$, where $C_r$ is the intersection of the open connected cone $C$ with neighbourhood of the origin of radius $r$. For all numbers $0 < \epsilon < 1$ and cone $C \subset\subset C$, let it satisfy the estimate

$$|f(x + iy)| \leqslant D_\epsilon(C')\{\epsilon\, M(ax) + \epsilon\, \Omega(b/y)\} \qquad (\ldots 5.1)$$

$z \in \mathbb{R}^n + iC_r$. Then $f(z)$ has a boundary value $f$ that belongs to the space $\left(W^{\Omega}_{M}\right)'$ to which it converges in the strong topology of this space as $y \rightarrow 0$ within an arbitrary cone that is compact in $C$.

Proof : Let $g$ be the spectral function of $f(z)$ on any space $\left(\widetilde{W}^{\Omega}_{M_1}\right) = W^{\Omega d_1}_{M_d}$ where $M < M_1$. In view of our assumption about the behaviour of $f(x + iy)$ as $y \rightarrow 0$ we shall show that the functional $g$ has growth order less than $-M_d(\tau/b)$. For $\psi \in \widetilde{W}^{\Omega}_{M1} = W^{\Omega d_1}_{M_d}$, the convolution

$$
\begin{aligned}
(g^* \psi)(\tau) &= < g(\sigma). \psi(\sigma+\tau)> \\
&= < g(\sigma)\, e^{-<\sigma \cdot y>}, \psi(\sigma+\tau)\, e^{<\sigma \cdot y>} > \\
&= < \mathscr{F}^{-1}[f(x+iy)], \mathscr{F}[\phi(x)\, e^{-i<\tau, x>}\, e^{<\sigma \cdot y>}]> \\
&= < \mathscr{F}^{-1}[f(x+iy)], \int_{\mathbb{R}^n} \phi(x)\, e^{-i<\tau \cdot x>}\, e^{<\sigma \cdot y>}\, e^{i<\sigma \cdot x>} dx > \\
&= < \mathscr{F}^{-1}[f(x+iy)], \mathscr{F}[\phi(x-iy)\, e^{-i<\tau \cdot x - iy>}]> \\
&= < f(x+iy), \phi(x-iy)\, e^{-i<\tau, x-iy>} >.
\end{aligned}
$$

Therefore,

$$
(g^* \psi)(\tau) = \int_{\mathbb{R}^n} f(x+iy_0)\, \phi(x-iy_0)\, e^{-i<\tau, x-iy_0>}\, dx \qquad \ldots(5.2)
$$

where $\phi(x) = \mathscr{F}^{-1}[\psi(\sigma)]$ and $y_0$ is an arbitrary point in $C_r$. We shall assume that $y_0 \in C'_r$ where $C'_r$ fixed cone that is compact in $C_r$. Now using (5.1) we have from (52) the estimate

$$
\begin{aligned}
|(g^*\psi)(\tau)| &\leqslant \int_{\mathbb{R}^n} |f(x+iy_0)|\ |\phi(x-iy_0)| \\
&\qquad \times e^{-<\tau, y_0>}\, |e^{-i<\tau \cdot x>}|\, dx \\
&\leqslant D_\epsilon(C') \int_{\mathbb{R}^n} (e^{\epsilon M(ax)+\epsilon \Omega(b/y_0)} \\
&\qquad \|\phi\|_{ab}\, e^{-M_1(ax)+\Omega(by_0)}\, e^{-<\tau, y_0>})dx \\
&= D_\epsilon(C')\, \|\phi\|_{ab} \exp\{\epsilon\, \Omega(b/y_0) + \Omega(by_0) - <\tau, y_0>\} \\
&\qquad \times \int_{\mathbb{R}^n} e^{-(1-\epsilon)M_1(ax)}\, dx \\
&= D'_\epsilon(C')\, \|\phi\|_{ab} \exp\left\{\epsilon\Omega\left(\frac{b}{y_0}\right) + \Omega\, by_0) - <\tau, y_0>\right\} \qquad \ldots(5.3)
\end{aligned}
$$

For any given $\tau$ we can choose $y_0 \in C'_r$ such that $<\tau, y_0> = M_a(\tau/b) + \Omega(by_0)$ holds. By this fact the estimate (5.3) becomes

$$
\begin{aligned}
|(g^* \psi)(\tau)| &\leqslant D_\epsilon'(C')\, \|\phi\|_{ab} \exp\{\epsilon\, \Omega(b/y_0) \\
&\qquad + \Omega(b\, y_0) - M_a(\tau/b) - \Omega(by_0)\} \\
&= D_\epsilon''(C')\, \|\phi\|_{ab} \exp\{-M_a(\tau/b)\}. \qquad \ldots(5.4)
\end{aligned}
$$

since $\Omega(b/y_0)$ is constant for a fixed $y_0$. Since the Fourier operator is continuous, then we have

$$
|(g^*\psi)(\tau)| \leqslant D_\epsilon'''(C')\, \|\psi\|_{ab} \exp\{-M_a(\tau/b)\} \qquad \ldots(5.5)
$$

which is the derived growth condition.

In view of Lemma 4.2, this behaviour of the functional $g$ at infinity means that it allows extension to the space $\widetilde{W}_M^{\mathfrak{a}} = W_{M_d}^{\mathfrak{a}_d}$ and it remains to show that the Fourier transform of this extension $f$ is the limit of the function $f(z)$ as $y \to 0$, $y \in C' \subset\subset C$, in the sense of convergence in the space $\left(\widetilde{W}_M^{\mathfrak{a}}\right)' = \left(W_{Md}^{\mathfrak{a}_d}\right)'$.

To prove this we note that the function

$$(g\, e^{-<\sigma, y>} * \psi)(\tau) = \int_{\mathbb{R}^n} f(x + iy_0)\, \phi\,(x - i\,(y_0 - y))$$

$$\exp^{-i<\tau, x - i(y_0 - y)>}\, dx \qquad \ldots(5.6)$$

satisfies the estimate (5.5) not only for $y = 0$ but also uniformly for $y \in \overline{C}_r'$ ; for if $-<\tau, y> \geqslant 0$, the estimate (5.5) can only improved and for $-<\tau, y> < 0$ setting $y = ry/\|y\|$, we can majorize this expression by a constant.

Now we shall show that the function $\int_{\mathbb{R}^n} <g\, e^{-<\sigma, y>}, \psi_0\,(\sigma + \tau)\, \psi\,(\sigma)> d\tau$ for $\psi \in \widetilde{W}_{M_1}^{\mathfrak{a}} = W_{M_d}^{\mathfrak{a}_{d_1}}$ and $\phi_0 \in \widetilde{W}_{M_0}^{\mathfrak{a}} = W_{Md}^{\mathfrak{a}_{d0}}$ $\int_{\mathbb{R}^n} \psi_0\,(\sigma)\, d\sigma = 1$, $M < M_1 < M_0$ is continuous with respect to $y$ in any sphere $|\,\psi\,| < 1$. Indeed $|<g\, e^{-<\sigma, y>}, \psi_0\,(\sigma + \tau)\, \psi\,(\sigma)>|$ can be written as $|<g\, e^{-<\sigma, y>}, \chi_r\,(\sigma + \tau)>|$, where $\chi_r\,(\sigma) = \psi_0\,(\sigma)\, \psi\, \sigma - \tau)$.

We have

$$|<g\, e^{-<\sigma, y>}, \chi_r\,(\sigma + \tau)>| \leqslant \|g\, e^{-<\sigma, y>}\|_{ab}\, \|\chi_r\|_{ab}\, e^{-M_d(ar)}.$$

From the proof of Lemma 4.2, it follows that $\|\chi_r\|_{ab} \leqslant L_1\, |\,\psi\,(\sigma)\,| \times e^{-M_d(ar)}$. Therefore, we have the estimate

$$|\,(g\, e^{-<\sigma, y>} * \chi_r)\,(\tau)\,| = |<g\, e^{-<\sigma, y>}, \chi_r\,(\sigma + \tau)>|$$

$$\leqslant \|g\, e^{-<\sigma, y>}\|_{ab}\, L_1\, |\,\psi\,(\sigma)\,|\, e^{-2M_d(a\epsilon)}.$$

By Proposition 4.6, $\|g\, e^{-<\sigma, y>}\|_{ab}$ is continuous in the domain $\overline{C}_r'$ . Since $|\,\psi\,|$ is continuous in $\overline{C}_r'$ , therefore $(g\, e^{-<\sigma, y>} * \chi_r)\,(\tau)$ is continuous in $\overline{C}_r'$ . Moreover $(g\, e^{-<\sigma, y>} * \chi_r)$ is uniformly continuous with respect to $y$ in any sphere $|\,\psi\,| < 1$. Therefore, the function $\int_{\mathbb{R}^n} < g\, e^{-<\sigma, y>} <, \psi_0\,(\sigma + \tau)\, \psi\,(\sigma)> d\tau$ is continuous in $\overline{C}_r'$ and moreover is uniformly continuous with respect to $y$ is any sphere $|\,\psi\,| < 1$. For $y \in C_r'$ this function is identical with $\int_{\mathbb{R}^n} f\,(x + iy)\, \phi\,(x)\, dx$. Hence, for all $a$ and $b$

$$\| f(x+iy) - f \|_{ab} = \sup_{\|\phi\|_{ab} \leqslant 1} \Big| \int_{\mathbb{R}^n} f(x+iy)\,\phi(x)\,dx$$

$$- \langle f\ \phi \rangle \Big| \to 0 \text{ where } y \to 0 \text{ within } C'.$$

This proves the theorem.

*Theorem* 5.8—Suppose the function $f(z)$ is holomorphic in the tube domain $T^C = \mathbb{R}^n + iC$, where $C$ is an open connected cone and for all number $0 < \epsilon < 1$ and a cone $C' \subset\subset C$ satisfies

$$| f(x+iy) | \leqslant (D_\epsilon (C') \exp \{\epsilon M(ax) + \epsilon\Omega(b/y) + (l+\epsilon+2)$$
$$\Omega(by)\}. \quad ..\ (5.7)$$

Then its spectral function $g \in \left(\widetilde{W}_M^{\Omega}\right)' = \left(W_{M_d}^{\Omega d}\right)'$ decreases not slower than

$$\exp\left\{-(l+\epsilon+1)\, M_d\left(\frac{\tau}{b(l+\epsilon+1)}\right)\right\} \text{ within any cone } C'_* \subset C_*$$

Conversely, any functional $g \in \left(\widetilde{W}_M^{\Omega}\right)' = \left(W_{M_d}^{\Omega d}\right)'$ which decreases not slower than $\exp\left\{-(l+\epsilon+1)\, M_d\left(\frac{\tau}{b(l+\epsilon+1)}\right)\right\}$ in $C_*$ has a Fourier-Laplace transform that is holomorphic in $T^C$ and satisfies the estimate (5.7).

Proof : Suppose $f(z)$ is holomorphic in $T^C$ and satisfies the estimate (5.7). Its spectral function $g \in \left(\widetilde{W}_M^{\Omega}\right)' = \left(W_{M_d}^{\Omega d}\right)'$ is obtained by Theorem 5.2, and our problem reduces to estimating the behaviour of $g$ in $C_*$. Let $\psi \in \widetilde{W}_{M_1}^{\Omega} = W_{M_d}^{\Omega d_1}$ with $M < M_1$. Then by definition of the spectral function,

$(g * \psi)(\tau) = \langle g(\sigma), \psi(\sigma+\tau) \rangle$ can be written in the form

$$(g * \psi)(\tau) = \int_{\mathbb{R}^n} f(x+iy_0)\,\phi(x-iy_0)\, e^{\langle i(\tau,x+iy_0)\rangle}\,dx \quad ..\ (5.8)$$

where $y_0 \in C' \subset\subset C$ and $\phi(x) = \mathscr{F}^{-1}[\psi(\sigma)]$. Now

$$|(g * \psi)(\tau)| \leqslant \int_{\mathbb{R}^n} |f(x+iy_0)||\phi(x-iy_0)|\, e^{-\langle \tau,y_0\rangle}\, |e^{-i\langle \tau,x\rangle}|\,dx$$

$$\leqslant D_\epsilon(C') \int_{\mathbb{R}^n} \exp\left\{\epsilon M(ax) + \epsilon\,\Omega\left(\frac{b}{y_0}\right)\right.$$

$$\left. + (l+\epsilon)\,\Omega(by_0)\right\} \|\phi\|_{ab}\, e^{-M_1(ax)+\Omega(by_0)}\; e^{-\langle \tau,y\rangle}\,dx$$

$$\leqslant D_\epsilon(C')\,\|\phi\|_{ab} \exp\{\epsilon\,\Omega(b/y_0) + (l+\epsilon+1)\,\Omega(by_0)$$

(*equation continued on p.* 1019)

$$- <\tau, y_0>\} \int_{\mathbb{R}^n} e^{-(1-\epsilon)M_1(ax)}\, dx$$

$$\leqslant D'_\epsilon\ (C')\, \|\psi\|_{ab} \exp\left\{\epsilon\, \Omega\left(\frac{b}{y_0}\right) + (l + \epsilon + I)\ \Omega\ (by_0) \right.$$

$$\left. - <\tau, y_0> \right\}. \qquad \text{..(5.9)}$$

For a given $\tau$, we can choose $y_0$ in such a way that $<\tau, y_0> \; = + M_d\ (\tau) + \Omega\ (y_0)$ holds. Therefore, $(l + \epsilon + 1)\ \Omega\ (by_0) - (\tau, y_0>$

$$= (l + \epsilon + 1) \left[\Omega\ (by_0) - \frac{<\tau, y_0>}{l + \epsilon + 1}\right]$$

$$= - (l + \epsilon + 1)\ M_d \left(\frac{\tau}{b\,(l + \epsilon + 1)}\right)$$

and for this fixed $y_0$, $e^{\epsilon\Omega(b/y_0)}$ is replaced by a constant. Hence the estimate (5.9) becomes

$$|\ (g^* \psi)\ (\tau)\ | \leqslant D^{*}_\epsilon\ (C')\, \|\psi\|_{ab} \exp\left\{- (l + \epsilon + 1)\ M_d \left(\frac{\tau}{b\,(l + \epsilon + 1)}\right)\right\} \qquad \text{...(5.10)}$$

which is the desired estimate.

Conversely, let $g \in \left(W^{\Omega}_{M}\right) = \left(W^{\Omega_d}_{M_d}\right)'$ and let it satisfy estimate (5.10) in $C_*$. Then its Fourier-Laplace transform $f(z) = \int_{\mathbb{R}^n} F(z, \tau)\, d\tau$ can be constructed using the auxiliary function $\psi_0 \in \tilde{W}^{\Omega}_{M} = W^{\Omega_{d_1}}_{M_d}$ with $M < M_1$. The problem is then reduced to estimating the behaviour of the function $F(z, \tau)$ which is defined by

$$F(z, \tau) = <g\,(\sigma), \psi_0\,(\sigma + \tau)\, e^{i<\sigma, z>}) > \qquad \text{...(5.11)}$$

with respect to $\tau$ for $z \in T^{C'}$. Then from the proof of Lemma (4.4) $f(z)$ is holomorphic in $T^C$.

Now we have

$$|\ F(z, \tau)\ | = |\ (g^* \psi_0\, e^{i<\sigma, z>})\ (\tau)\ |\ e^{+<\tau, y>}. \qquad \text{...(5.12)}$$

Suppose $\tau$ varies within a cone $C'_* \subset\subset C_*$. Then our assumption concerning the behavior of $g$ in $C_*$ gives

$$|\ F(z, \tau)\ | \leqslant D_\epsilon\ (C')\, \|\psi_0\ e^{i<\sigma, z>}\|_{ab}\ \exp\{- (l + \epsilon + 1)\ M_d$$

$$\left(\frac{\tau}{b\,(l + \epsilon + 1)}\right) + <\tau, y>\Big\}. \qquad \text{...(5.13)}$$

Assuming $C'_* \subset\subset C_*$ we can write that for all $z \in T^{C'}$, $C' \subset\subset C$, $\tau \in \mathbb{R}^n \setminus C_*$, the estimate

$$| F(z, \tau) | \leqslant D_\varepsilon(C') \| \psi_0 e^{i<\sigma, z>} \|_{ab} \exp\{-M_a(\tau/b) + <\tau, y>\}. \quad \ldots(5.14)$$

The inequality (5.14) ensures convergence of the integral $f(z) = \int_{\mathbb{R}^n} F(z, \tau)\, d\tau$ and leads to the following estimate for the Fourier-Laplace transform of $f(z)$.

$$| f(z) | \leqslant D_\varepsilon(C') \| \psi_0 e^{i<\sigma, z>} \|_{ab} \times [\sup_{\tau \in \mathbb{R}^n \setminus C_*} \exp\{-M_a(\tau/b) + <\tau, y>\}$$

$$+ [\sup_{\tau \in \mathbb{R}^n C'_*} \exp\left\{-(l+\epsilon+1)\, M_a\left(\frac{\tau}{b(l+\epsilon+1)}\right)\right.$$

$$\left. + <\tau, y>\right\}].$$

By the Young Inequality, we have

$$-M_a(\tau/b) + <\tau, y> \leqslant -M_a(\tau/b) + M_a(\tau/b) + \Omega(by)$$

and

$$-(l+\epsilon+1)\, M_a\left(\frac{\tau}{b(l+\epsilon+1)}\right) + <\tau, y>$$

$$= -(l+\epsilon+1)\, M_a\left(\frac{\tau}{b(l+\epsilon+1)}\right) + <\frac{\tau}{b(l+\epsilon+1)},$$

$$b(l+\epsilon+1)\, y>$$

$$\leqslant -(l+\epsilon+1)\, M_a\left(\frac{\tau}{b(l+\epsilon+1)}\right) + (l+\epsilon+1)$$

$$\times \left\{ M_a\left(\frac{\tau}{b(l+\epsilon+1)}\right) + \Omega(by) \right\}$$

$$= (l+\epsilon+1)\, \Omega(by).$$

With regard to the factor $\| \psi_0 e^{i<\sigma, z>} \|_{ab}$ the continuity of the Fourier operator enables us to replace it by $\| \phi_0(\xi + z) \|_{ab}$; and hence

$$\| \phi_0(\xi + x + iy) \|_{ab}$$

$$\leqslant \sup | \phi_0(\xi + x + iy) | \; e^{M(a\xi) - \Omega(by)}$$

$$\leqslant \sup | \phi_0(t + iy) | \; e^{M(at) - \Omega(by)}\, e^{-M(ax)}$$

$$= \| \phi_0 \|_{ab}\, e^{-M(ax)}$$

$$\leqslant \| \phi_0 \|_{ab}\, e^{\epsilon M(ax)}.$$

Therefore, (5.15) reduces to

$$|f(z)| \leqslant D_\varepsilon(C') \exp\{\epsilon M(ax) + (l + \epsilon + 2)\,\Omega(by)\}$$
$$\leqslant D_\varepsilon(C') \exp\{\epsilon M(ax) + (l + \epsilon + 2)\,\Omega(by) + \epsilon\Omega(b/y)\}$$

since

$$1 \leqslant e^{\epsilon\Omega(b/y)}.$$


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# THE AXISYMMETRIC CAUCHY-POISSON PROBLEM IN A STRATIFIED LIQUID

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An initial value investigation is made of the axisymmetric free surface response of a non-rotating stratified liquid of fiuite or infinite depth to the initial displacement of the free surface. Based upon the Boussinesq approximation, the problem is solved by the joint Laplace and Hankel transforms and the integral solution is obtained. It is shown that the solution represents the linear superposition of the dispersive waves. Special attention is given to the governing dispersion relations corresponding to small or large frequency and the Brunt Vaisala frequency parameter. The asymptotic analysis of the integral solution is carried out by the stationary phase method to describe the solution for large time and distance from the origin of disturbance. It is shown that the asymptotic solution consists of the classical free surface gravity waves and the internal waves.

## 1. INTRODUCTION

Historically, Cauchy (1789-1857) and Poisson (1781-1840) first initiated the study of surface waves produced in deep water by a local disturbances on the free surface. These studies were published independently in two classical memoirs by Poisson[7] and Cauchy[2]. Subsequent investigations into the classical Cauchy-Poisson wave problem for an inviscid liquid was continued by many others including Lord Kelvin[4], Burnside[1] and Lamb[5]. In order to determine solutions of the linearized Cauchy-Poisson problem by a simpler method, Kelvin[4] developed the method of stationary phase and applied it to obtain the asymptotic solution of the wave motions due to concentrated surface elevation in deep water. Some of these solutions associated with various physical and/or geometric configurations are available in Lamb's treatise on Hydrodynamics[6].

In recent years, several authors including Debnath[3], Nikitin and Podrezov[8] and Nikitin and Potetyunko[9] have given considerable attention to the Cauchy-Poisson problems in an incompressible homogeneous inviscid or viscous liquid. The two- or three-dimensional or axisymmetric free surface response of the liquid to either initial displacement of the free surface or an initial pressure impulse applied to the free surface has also been determined by asymptotic methods. In spite of these studies, the Cauchy-

Poisson wave problem in a non-rotating stratified liquid has not yet been investigated. The main purpose of this paper is to study such a problem which seems to be important and interesting from a mathematical as well as a physical point of view.

This paper is concerned with the initial value investigation of the axisymmetric free surface response of a non-rotating stratified liquid of finite or infinite depth. Based upon the Boussinesq approximation, the problem is solved by the joint use of the Laplace and Hankel transforms. The integral solution seems to be the linear superposition of the dispersive waves. Special attention is given to the governing dispersion relations corresponding to small or large the Brunt Vaisala frequency parameter. The asymptotic analysis of the integral solution is carried out by the stationary phase approximation to determine the solution for large time and distance from the source of the disturbance. The asymptotic solution is found to consist of the classical free surface gravity waves and the internal waves.

## 2. Mathematical Formulation of the Problem

We consider the axisymmetric Cauchy-Poisson wave problem in an inviscid incompressible stratified liquid of finite or infinite depth. The problem will be studied under the Boussinesq approximation and the density field of the undisturbed liquid is assumed to be in the form

$$\rho_0 = \rho_{00} \exp(-\beta z), \ \beta > 0 \qquad \ldots(2.1)$$

where $\rho_{00}$ and $\beta$ are constants. The Brunt Vaisala frequency $N$ is defined by

$$N = \left\{- \frac{g}{\rho_0} \frac{d\rho_0}{dz}\right\}^{1/2} \qquad \ldots(2.2)$$

which is real when the mean density distribution is stable ($d\rho_0/dz < 0$) and it remains constant throughout the flow field because of (2.1).

With the cylindrical polar coordinates $(r, \theta, z)$, we consider a semi-infinite body of liquid bounded by $0 \leqslant r < \infty$ and $-h < z \leqslant 0$. The equilibrium state to be perturbed is the state of rest so the distribution of pressure and density is the hydrostatic equilibrium state given by $p = p_0(z)$ and $\rho = \rho_0(z)$ with $d\rho_0 = \rho_0\, gdz$ where g is the acceleration due to gravity. Invoking the axial symmetry and linearization, the velocity components without swirling motion ($v = 0$), $u$, $w$ and the small perturbation and pressure $p'$ are governed by the equations of motion

$$\rho_0 \frac{\partial}{\partial t}(u, w) = -\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right) p' - (0, g\rho') \qquad \ldots(2.3ab)$$

where $\rho'$ is the density perturbation.

In view of the Boussinesq approximatian, we take $\rho_0$ as constant in (2.3ab) and introduce the acceleration potential $\chi \equiv p'/\rho_0 + gz$ and we can rewrite (2.3ab) in terms of $\chi$ as

$$\frac{\partial}{\partial t}(u, w) = -\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)(\chi - gz) - (0, g\rho'/\rho_0). \quad \text{...(2.4ab)}$$

The condition of incompressibility of the liquid is

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0. \quad \text{...(2.5)}$$

The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \quad \text{...(2.6)}$$

Introducing the free surface displacement function $\eta(r, t)$, the free surface boundary conditions are

$$\chi = g\eta, w = \eta_t \qquad \text{on } z = 0, t > 0. \quad \text{...(2.7ab)}$$

The bottom boundary condition is

$$\frac{\partial \chi}{\partial z} \to 0 \text{ as } z \to -h \text{ or } z \to -\infty. \quad \text{...(2.8)}$$

We further assume that $\chi$ and its derivations vanish as $r \to \infty$ and/or $z \to -h$.

The flow is generated in the liquid by the action of the free surface displacement at $t = 0$ so that the initial conditions are

$$u = v = \chi = 0, \eta(r, t) = \eta_0(r) \quad \text{at } t = 0, 0 \leqslant r < \infty. \quad \text{...(2.9)}$$

We further assume that $\eta_0(r)$ is sufficiently small so that the problem can be treated within the scope of the linearized theory.

## 3. The Solution of the Problem and the Dispersion Relation

It is convenient to reduce the initial value problem governed by the differential system (2.4ab) — (2.10ab) to a boundary value problem by using the Laplace transform

$$\bar{\chi}(r, z, s) = \int_0^\infty e^{-st}\chi(r, z, t)\,dt. \quad \text{...(3.1)}$$

In view of this transformation, the system (2.4ab) — (2.10ab) reduces to the boundary value problem

$$\bar{\chi}_{rr} + \frac{\bar{\chi}_r}{r} + \lambda^2\bar{\chi}_{zz} = 0, 0 \leqslant r < \infty, -h \leqslant z \leqslant 0 \quad \text{...(3.2)}$$

where

$$\lambda^2 = \frac{s^2}{s^2 + N^2} \quad \text{...(3.3)}$$

$$\bar{\chi} = g\bar{\eta}, \quad \frac{s^2}{g}\bar{\chi} + \lambda^2 \bar{\chi}_z = s\eta_0(r) \qquad \text{on } z = 0 \qquad \ldots(3.4ab)$$

$$\bar{\chi}_z \to 0 \text{ as } z \to -h \text{ or } z \to -\infty. \qquad \ldots(3.5)$$

To solve the boundary value problem (3.2) – (3.5) we apply the Hankel transformation of zero order

$$\tilde{\bar{\chi}}(k, z, s) = \int_0^\infty r\, J_0(kr)\, \bar{\chi}(r, z, s)\, dr. \qquad \ldots(3.6)$$

In view of this transformation, the system (3.2) – (3.5) reduces to the form

$$\tilde{\bar{\chi}}_{zz} = \frac{k^2}{\lambda^2}\tilde{\bar{\chi}}, \; -h \leqslant z \leqslant 0 \text{ or } -\infty < z \leqslant 0 \qquad \ldots(3.7)$$

$$\tilde{\bar{\chi}} = g\tilde{\bar{\eta}}, \; \frac{s^2}{g}\tilde{\bar{\chi}} + \lambda^2 \tilde{\bar{\chi}}_z = s\,\tilde{\eta}_0(k) \text{ on } z = 0 \qquad \ldots(3.8ab)$$

$$\tilde{\bar{\chi}}_z \to 0 \text{ as } z \to -h \text{ or } z \to -\infty. \qquad \ldots(3.9)$$

The solutions for $\tilde{\bar{\chi}}$ and $\tilde{\bar{\eta}}$ in a liquid of finite depth are

$$\tilde{\bar{\chi}}(k, z, s) = \frac{g\, s\, \tilde{\eta}_0(k) \cosh \frac{k}{\lambda}(z+h)}{s^2 \cosh\left(\frac{kh}{\lambda}\right) + gk\lambda \sinh\left(\frac{kh}{\lambda}\right)} \qquad \ldots(3.10)$$

$$\tilde{\bar{\eta}}(k, s) = \frac{s\, \tilde{\eta}_0(k)}{s^2 \cosh\left(\frac{kh}{\lambda}\right) + gk\lambda \sinh\left(\frac{kh}{\lambda}\right)}. \qquad \ldots(3.11)$$

The inverse Laplace-Hankel transforms gives the solutions in a liquid of finite depth

$$\chi(r, z, t) = \frac{g}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\, e^{st}\, ds \int_0^\infty \frac{\tilde{\eta}_0(k)\, k\, J_0(kr) \cosh \frac{k}{\lambda}(z+h)\, dk}{s^2 \cosh\left(\frac{kh}{\lambda}\right) + gk\lambda \sinh\left(\frac{kh}{\lambda}\right)} \qquad \ldots(3.12)$$

$$\eta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\, e^{st}\, ds \int_0^\infty \frac{\tilde{\eta}_0(k)\, k\, J_0(kr)\, dk}{s^2 \cosh\left(\frac{kh}{\lambda}\right) + gk\lambda \sinh\left(\frac{kh}{\lambda}\right)} \qquad \ldots(3.13)$$

The solutions for $\tilde{\bar{\chi}}$ and $\tilde{\bar{\eta}}$ in a liquid of infinite depth are given

$$\tilde{\bar{\chi}}\ (k, z, s) = \frac{sg\ \tilde{\eta}_0\,(k)}{s^2 + gk\lambda}\, e^{kz/\lambda} \qquad \text{...(3.14)}$$

$$\tilde{\bar{\eta}}\ (k, s) = \frac{s\ \tilde{\eta}_0\,(k)}{s^2 + gk\lambda}. \qquad \text{...(3.15)}$$

Application of the inverse Laplace-Hankel transformations gives

$$\chi = \frac{g}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\ e^{st}\,ds\int_0^\infty \frac{\tilde{\eta}_0\,(k)}{(s^2+gk\lambda)}\exp\left(\frac{k\,z}{\lambda}\right)kJ_0\,(kr)\,dk \qquad \text{...(3.16)}$$

$$\eta = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\ e^{st}\,ds\int_0^\infty \frac{k\tilde{\eta}_0\,(k)\,J_0\,(kr)\,dk}{(s^2+gk\lambda)} \qquad \text{.. (3.17a)}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(s+\frac{N^2}{s}\right)e^{st}\,ds\int_0^\infty \frac{k\ \tilde{\eta}_0\,(k)\,J_0\,(kr)\,dk}{\left(s^2+N^2+\dfrac{gk}{\lambda}\right)} \qquad \text{...(3.17b)}$$

where Re $(\lambda) > 0$ that is necessary for a meaningful dispersion relation.

We guarantee the restriction Re $(\lambda) > 0$ imposed in solutions (3.16) — (3.17) by selecting an appropriate branch cut along Im $(s)$ between the branch points at $s = \pm iN$. It is easy to check that the integrand in (3.17) has two imaginary poles at $s = \pm i\omega$ outside the branch cut $(\omega > N)$. Consequently, substituting $s = \pm\ iN$.
$\lambda^2 = (1 - N^2\,\omega^{-2})$ in the denominator of the integrand of the solutions and equating the resulting denominator to zero, we obtain the implicit dispersion relation

$$\omega^2 = N^2 + \frac{gk}{\lambda}\ ,\ \lambda = \left(1-\frac{N^2}{\omega^2}\right)^{-1/2}. \qquad \text{...(3.18ab)}$$

Eliminating of $\lambda$ from (3.18ab) and expanding of this result for small $N^2$, we obtain

$$\omega^2 = \frac{N^2}{2} + g\,k + O\left(\frac{N^4}{4g^2\,k^2}\right)\ \text{as}\ N^2\to 0. \qquad \text{...(3.19)}$$

In the limit of $N^2 \to 0$, (3.18ab) yields the well known dispersion relation $\omega^2 = g\,k$ for the deep water waves in the absence of stratification.

## 4. Asymptotic Analysis of the Free Surface Displacement Function

In the absence of stratification ($N = 0$), the initial value problem formulated in previous sections reduces to the classical Cauchy-Poisson problem which has the solu-

tion (Lamb[6], p. 432) in the similarity form

$$\eta(r, t) = \frac{1}{\sqrt{2\pi r^2}} \xi F(\xi), \; \xi = \frac{g t^2}{4r} \qquad \text{...(4.1ab)}$$

where $F(\xi) = \cos \xi$.

Invoking $\eta_0(r) = a\, \delta(r)/r$ so that $\tilde{\eta}_0(k) = a$, and introducing non-dimensional free-surface displacement function $\eta^* = r^2/a\, \eta(r, t)$, we obtain from (3.17b) that

$$\eta^*(r, t) = \frac{r^2}{2\pi i} \int_{-i\infty}^{i\infty} \left(s + \frac{N^2}{s}\right) e^{st}\, ds \int_0^{\infty} \frac{k\, J_0(kr)\, dk}{\left(s^2 + N^2 + \frac{g k}{\lambda}\right)} \qquad \text{...(4.2)}$$

where the path of integration in the complex $s$-plane passes to the right of both the poles at $s = \pm i\omega$ and the branch cut between $s = \pm iN$, $\omega > N$.

It is, in principle, possible to develop $\eta^*(r, t)$ as a power series in time $t$ by expanding its Laplace transform function about $s = \infty$. The power series representation is quite complicated compared to the classical solution given in Lamb (1932), and it does not admit to a wave interpretation. Consequently, we write the solution only to the first approximation

$$\eta^*(r, t) = r^2 \int_0^{\infty} k\; J_0(kr)\, kd\, \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [\left(\frac{1}{s} - \frac{gk}{s^3}\right) + O\left(\omega_0^2\, s^{-5}, N^2 \omega_0 \right.$$

$$\left. \times\, s^{-5}\right)]\, e^{st}\, ds \qquad \text{...(4.3)}$$

where $\omega_0^2 = gk$.

We next introduce non-dimensional quantities defined by

$$\kappa \equiv \frac{g t^2}{4r}, \; \alpha \equiv \frac{N^2 r}{g}, \; \tau = (\kappa\lambda)^{1/2} = Nt \qquad \text{...(4.4abc)}$$

a change of variable $kr = \zeta$ and then perform term by term Laplace inversion to obtain the solution

$$\eta^*(r, t) = \int_0^{\infty} [1 - 2\kappa\zeta + O\{\kappa^2 \zeta^2, \kappa\, \zeta\tau^2\}]\, \zeta\, J_0(\zeta)\, d\zeta. \qquad \text{...(4.5)}$$

This integral can be evaluated by using standard integrals of Bessel functions to obtain

$$\eta^*(r, t) = \frac{2\Gamma(1)}{\Gamma(0)} - 2\kappa \frac{2^2\, \Gamma(\frac{1}{2} + 1)}{\Gamma(-\frac{1}{2})} + O(\kappa^2, \kappa\tau^2)$$

$$\sim \frac{g t^2}{2r} + O(\kappa^2, \kappa\tau^2). \qquad \text{...(4.6)}$$

This corresponds to classical similarity from of the solution with $gt^2/r$ as the similarity variable.

In order to determine the wave structure in the stratified liquids, the asymptotic solution for sufficiently large time is of special interest. It is then convenient to deform the path of integration of the $s$-integral in (4.2) into three closed loops separately encircling the poles and the branch cut. At the same time, we identify the combined contribution of the poles as $\eta_1^*$ and the contribution of the branch points as $\eta_2^*$. We next apply the laplace inversion to write down the integral solution for as $\eta_1^*$ in the form

$$\eta_1^* (r, t) = r^2 \int_0^\infty \cos \omega t \, J_0 (kr) \, kdk \qquad \ldots(4.7)$$

where $\omega^2 = N^2 + \frac{gk}{\lambda}$ which becomes, in the limit $N^2 \to 0$

$$\omega^2 \sim g\,k + \frac{N^2}{2} + O(N^4). \qquad \ldots(4.8)$$

In view of this limiting expression combined with a change of variable $kr = \xi/\lambda$ we obtain

$$\eta_1^* = \int_0^\infty \frac{\cos \omega t}{\lambda^2} J_0 \left( \frac{\xi}{\lambda} \right) \xi \, d\, \xi \qquad \ldots(4.9)$$

$$= \int_0^\infty \frac{1}{\lambda^2} \cos [(4\kappa)^{1/2} \left( \xi + \frac{N^2 r}{g} \right)^{1/2} ] \, J_0 [\xi + \frac{N^2 r}{2g} + O\left( \frac{N^2 r}{g} \times \xi^{-1} \right)] \, \xi \, d\, \xi \qquad \ldots(4.10)$$

where

$$\omega^2 = N^2 \frac{g\xi}{r}. \qquad \ldots(4.11)$$

The exact evaluation of the integral in (4.10) is a formidable task. Hence the stationary phase method is employed to determine the dominant contribution of the integral for $4\kappa \gg 1, \alpha$. we then replace $J_0(z)$ by its asymptotic formula

$$J_0 (z) \sim \left( \frac{2}{\pi z} \right) \cos \left( z - \frac{\pi}{4} \right) z \to \infty. \qquad \ldots(4.12)$$

In the integral (4.10) to obtain

$$\eta_1^* = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty [\sqrt{\xi} + \frac{3}{2}\frac{N^1 r}{g\sqrt{\xi}} + O(\xi^{-3/2}, \frac{N^4 r^2}{g}\xi^{-3/2}))$$

$$X \cos[\sqrt{4\kappa}\left(\xi + \frac{N^2 r)}{g}\right)^{1/2}] \cos[\xi + \frac{N^2 r}{2g} - \frac{\pi}{4}]d\xi. \quad \ldots(4.13)$$

We next express the product of cosines by the sum of the cosines of the sum and difference of the two arguments so that (4.13) becomes

$$\eta_1^* \sim \frac{1}{\sqrt{2\pi}}\int^\infty \left(\sqrt{\xi} + \frac{3\alpha}{2\sqrt{\xi}}\right)[\cos\{(4\kappa)^{1/2}(\xi+\alpha)^{1/2} + \xi + \frac{\alpha}{2}$$

$$- \frac{\pi}{4}\} + \cos\{(4\kappa)^{1/2}(\xi+\alpha)^{1/2} - \xi - \frac{\alpha}{2} + \frac{\pi}{4}\}]\,d\xi. \quad \ldots(4.14)$$

We next apply the stationary phase method for large $4\kappa$ to obtain significant contributions of the two integrals in (4.14) to $\eta_1^*$. The first integral does not have any stationary point and so it has no contributions. But the second integral has a stationary point at $\xi_1 = \kappa - \alpha$ which makes a significant to this integral for large $4\kappa$. Hence evaluation of the contribution from $\xi_1 = \kappa - \alpha$ to the integral gives

$$\eta_1^* \sim \left(\sqrt{\xi_1} + \frac{3\alpha}{2\sqrt{\xi_1}}\right)(2\kappa)^{1/2}\cos\left(\kappa + \frac{\alpha}{2}\right). \quad \ldots(4.15)$$

In terms of the original variables with $4\kappa > \alpha$, this result has the representation

$$\eta_1^* \sim \frac{1}{\sqrt{2}}\left(\frac{gt^2}{2r} + \frac{N^2 r}{g}\right)\left(\frac{gt^2}{4r} - \frac{N^2 r}{g}\right)^{-1/2}\left(\frac{gt^2}{4r}\right)^{1/2}\cos$$

$$\times\left(\frac{gt^2}{4r} + \frac{N^2 r}{2g}\right). \quad \ldots(4.16)$$

This result corresponds to the surface waves in a stratified liquid governed by the modified dispersion relation (3.14), and is believed to be new. In the limit $N \to 0$, the result (4.16) agrees with the classical solution of the Cauchy-Poisson problem in a non-stratified liquid.

We next turn our attention to the evaluation of $\eta_2^*$ and make some changes of variables

$$s = iN\sin\theta,\ \lambda = -i\tan\theta,\ kr = \zeta \quad \ldots(4.17abc)$$

in (4.2) to obtain

$$\eta_2^* = \frac{N^2}{2\pi}\int_0^{2\pi}\exp(iNt\sin\theta)\cos\theta\sin\theta\,d\theta\int_0^\infty \frac{\zeta_0 J(\zeta)\,d\lambda}{iN^2\sin\theta\cos\theta - g\zeta/r} \quad \ldots(4.18)$$

where Im $\theta \to -$.

A simple transformation of contributions of each of the four $\theta$-quadrants to the first quadrant combined with Im $(\theta) \to 0 -$ gives

$$\eta_2^* = \frac{4N^2}{2\pi} \int_0^{\pi/2} \cos (Nt \sin \theta) \cos^2 \theta \, d\theta \int_0^{\infty} \frac{\zeta J_0(\zeta) \, d\zeta}{iN^2 \sin 2\theta - 2g\zeta/r^{-1}} \quad \ldots(4.19)$$

After some algebraic manipulation of the integrand of the $\zeta$-integral, it turns out that the integrand is of the order $O(\alpha^2)$ and hence the $\zeta$-integral has no significant contributions. The final result can be written in the form

$$\eta_2^* \sim - \frac{2N^2 r}{\pi g} \int_0^{\pi/2} \cos (Nt \sin \theta) \cos^2 \theta \, d\theta + O(\alpha^2)$$

$$= - \frac{2N^2 r}{\pi g} \frac{1}{Nt} J_1(Nt) + O(\alpha^2) \quad \ldots(4.20)$$

$$\sim - \frac{2N^2 r}{\pi g (Nt)} \left( \frac{2}{\pi Nt} \right)^{1/2} \sin \left( Nt - \frac{\pi}{4} \right) \text{ as } Nt \to \infty$$

$$= - \alpha \left( \frac{2}{\pi Nt} \right)^{3/2} \sin \left( Nt - \frac{\pi}{4} \right) \text{ as } Nt \to \infty. \quad \ldots(4.21)$$

This asymptotic result corresponds to waves of amplitude $O(\alpha (Nt)^{-3/2})$ and frequency $N$. The existence of these waves is entirely due to the density stratification, and has no antecedents in a non-stratified liquid. Ultimately, these waves will decay as $Nt \to \infty$.

## 5. Discussions and Conclusions

It follows from the above asymptotic analysis that the dimensional form of the free surface elevation has the following asymptotic representation

$$\eta(r, t) = \frac{a}{r^2} \eta^*(r, t)$$

$$= \frac{a}{r^2} [\eta_1^*(r, t) + \eta_2^*(r, t)]$$

$$\sim \frac{a}{\sqrt{2} r^2} \left( \frac{gt^2}{2r} + \frac{rN^2}{g} \right) \left( \frac{gt^2}{4r} - \alpha \right)^{-1/2} \left( \frac{gt^2}{4r} \right)^{1/2}$$

$$\cos \left( \frac{gt^2}{4r} + \alpha \right) - \frac{a\alpha}{r^2} \left( \frac{2}{\pi Nt} \right)^{3/2} \sin \left( Nt - \frac{\pi}{4} \right). \quad \ldots(5.1)$$

This solution consists of two distinct terms representing waves. The first term in (5.1) corresponds to surface waves which are qualitatively similar to those in the

classical Cauchy-Poisson problem for an inviscid non-stratified liquid. The amplitude of the surface waves is modified by stratification. But the principal effect of stratification on the classical solution is the phase shift by an amount $\alpha$ in the asymptotic wavetrains. The second term in (5.1) also represents waves of frequency $N$ and amplitude $O\,(\alpha r^{-2}\,Nt)^{-3/2})$ which decays to zere as $Nt \to \infty$. These are not surface waves and their existence is eniirely due to the density stratification. They have no antecedents in a non-stratified inviscid liquid.

As an example, we mention the initial displacement produced by a surface depression in the form

$$\eta_0\,(r) = \eta_0\,[1 - \left(\frac{r}{a}\right)^2]\,\exp\left(-\,\frac{r^2}{a^2}\right) \qquad \ldots(5.2)$$

where the surface depression is in the form of a cavity of constant depth $\eta_0$ with a concentric lip. The Hankel transform of $\eta_0\,(r)$ is

$$\tilde{\eta}_0\,(k) = \tfrac{1}{8}\,\eta_0\,a^2\,(ka)^2\,\exp\,[-\,\tfrac{1}{4}\,(ka)^2]. \qquad \ldots(5.3)$$

As asymptotic analysis similar to the case of the initially concentrated surface disturbance can be carried out without any difficulty. Except the mathematical form of the solution, the solution and its essential features will remain unchanged. Hence further discussion of this case would bs redundant.

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# ON TRANSIENT DEVELOPMENT OF WAVES AT AN INTERFACE BETWEEN TWO FLUIDS

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The transient gravity interface waves generated by a harmonically oscillating wavemaker immersed vertically in two incompressible fluids is considered. The resulting linearized initial value problem is solved using the method of generalized functions, and asymptotic analysis for large time and distance is given for the interface elevation.

## 1. INTRODUCTION

The two-dimensional problem of gravity waves generated by moving oscillating surface distributions in a fluid which is unbounded in both horizontal directions has been studied by Kaplan[1] and Debnath and Rosenblat[2] in infinite depth and in finite depth respectively. Pramanik[3] considered the initial value problem of waves generated by a moving oscillating surface pressure against a vertical cliff and a uniform asymptotic analysis was given for the unsteady case. Debnath and Basu[4] treated the same problem taking into account the effect of surface tension. Faltas[5] investigated the initial value problem of surface waves generated by a harmonically oscillating vertical wavemaker immersed in an infinite incompressible fluid of finite constant depth. It is the purpose of this paper to discuss the transient development of two-dimensional linearised waves at the interface between two fluids. The waves are produced by a harmonically oscillating wavemaker immersed vertically in both fluids. The integral representation of interface elevation is obtained through an application of the Laplace and the generalized cosine Fourier transforms of the equations of motion. Then the application of the stationary phase method combined with the contour integration method leads to the asymptotic waves valid for large time and distance.

## 2. FORMULATION OF THE PROBLEM AND FORMAL SOLUTION

We are concerned with the transient development of two dimensional infinitesimal wave motion of two superposed immiscible, non-viscous and incompressible fluids separated by a common interface. The waves are generated by a harmonically oscillating wavemaker immersed vertically in the two fluids. If the motion is generated originally from rest by the oscillations of the wavemaker; it will be irrotational throughout all time and we may describe the motion in the lower and upper fluids in terms of

velocity potentials $\phi\ (x, y, t)$, $\phi'\ (x, y, t)$ respectively. Take the origin $O$ at the mean level of the interface and the axis $Oy$ points vertically downwards along the wavemaker, the upper fluid being of infinite depth while the lower one is of finite constant depth $h$. The unsteady motions are produced in the two fluids by the continuous oscillations of the wavemaker, let it oscillate horizontally with velocity $U\ (y, t)$.

The velocity potentials satisfy an initial boundary-value problem in which

$$\nabla^2 \phi = 0,\ 0 < x < \infty,\ 0 < y < h,\ t > 0 \qquad \text{...(2.1)}$$

$$\nabla^2 \phi' = 0,\ 0 < x < \infty,\ -\infty < y < 0,\ t > 0 \qquad \text{...(2.2)}$$

with the bottom condition

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = h,\ t > 0; \qquad \text{...(2.3)}$$

also

$$\frac{\partial \phi'}{\partial y} \to 0 \text{ as } y \to -\infty,\ t > 0. \qquad \text{...(2.4)}$$

Neglecting surface tension, the linearised pressure condition is

$$\frac{\partial \phi}{\partial t} - s\ \frac{\partial \phi'}{\partial t} = g\ (1 - s)\ \eta, \text{ on } y = 0,\ t > 0, \qquad \text{...(2.5)}$$

where $s\ (0 < s < 1)$ is the ratio of the densities of the upper and lower fluids, $\eta = \eta\ (x, t)$ is the elevation of the interface above its mean level and $g$ is the acceleration due to gravity. The linearised kinematical boundary condition at the interface is

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi'}{\partial y} = \frac{\partial \eta}{\partial t},\ y = 0,\ t > 0. \qquad \text{...(2.6)}$$

At the wavemaker

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= U\ (y, t),\ 0 < y < h, && \text{...(2.7)} \\ \frac{\partial \phi'}{\partial x} &= U\ (y, t);\ -\infty < y < 0 && \text{...(2.8)} \end{aligned}\right\} \text{ on } x = 0,\ t > 0$$

and the initial conditions are

$$\phi = \phi' = \eta = 0, \text{ when } t = 0. \qquad \text{...(2.9)}$$

Also we suppose that $\phi$, $\phi'$, $\eta$ are defined in the generalised sense of Lighthill[6].

We introduce the Fourier cosine transform with respect to $x$ and the Laplace transfrom with respect to $t$ as

$$F_c\ (k, y, r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos\ kx\ dx \int_0^\infty e^{-rt}\ F\ (x, y, t)\ dt \qquad \text{..(2.10)}$$

where the suffix $c$ and the bar in the transformed function refer to the consine Fourier and Laplace transforms respectively.

Application of (2.10) to the system (2.1) — (2.9) gives

$$\frac{d^2}{dy^2}\bar{\phi}_c - k^2\bar{\phi}_c = \sqrt{\frac{2}{\pi}}\,\bar{U}(y, r) \quad , r > 0 \qquad \ldots(2.11)$$

$$\frac{d^2}{dy^2}\bar{\phi}'_c - k^2\bar{\phi}'_c = \sqrt{\frac{2}{\pi}}\,\bar{U}(y, r), r > 0 \qquad \ldots(2.12)$$

$$\frac{d}{dy}\bar{\phi}_c = 0, \text{ on } y = h, r > 0 \qquad \ldots(2.13)$$

$$\frac{d}{dy}\bar{\phi}'_c \to 0 \text{ as } y \to -\infty, r > 0 \qquad \ldots(2.14)$$

$$\left.\begin{aligned} &\bar{\phi}_c - s\,\bar{\phi}'_c = \frac{g}{r}(1 - s)\,\bar{\eta}_c \\ &\frac{d}{dy}\bar{\phi}_c = \frac{d}{dy}\bar{\phi}'_c = r\,\bar{\eta}_c \end{aligned}\right\} \text{ on } y = 0,\ r > 0. \qquad \ldots(2.15)$$

The solution of (2.11) is

$$\bar{\phi}_c(k, y, r) = A(k, r)\,e^{ky} + B(k, r)\,e^{-ky}$$

$$+ \sqrt{\frac{2}{\pi}}\int_0^y k^{-1}\sinh k(y - z)\,\bar{U}(z, r)\,dz,\ y > 0 \qquad \ldots(2.16)$$

while the solution of (2.12), satisfying condition (2.14) is

$$\bar{\phi}'_c(k, y, r) = C(k, r)\,e^{ky} + \tfrac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{-\infty} k^{-1}\,e^{k(z - y)}\,\bar{U}(z, r)\,dz$$

$$+ \sqrt{\frac{2}{\pi}}\int_0^y k^{-1}\sinh k(y - z)\,\bar{U}(z, r)\,dz,\ y > 0 \quad \ldots(2.17)$$

where $A(k, r)$, $B(k, r)$ and $C(k, r)$ are functions to be determined.

The transformed boundary conditions (2.15) are satisfied if

$$\left.\begin{aligned}
A(k, r) &= \frac{1}{2kr}[kg(1-s)+(1+s)r^2]\bar{\eta}_c + \frac{s}{2}\sqrt{\frac{2}{\pi}}\int_0^{-\infty} k^{-1} e^{kz} \\
&\qquad \times \bar{U}(z, r)\, dz \\
B(k, r) &= \frac{1}{2kr}(1-s)(kg-r^2)\bar{\eta}_c + \frac{s}{2}\sqrt{\frac{2}{\pi}}\int_0^{-\infty} k^{-1} e^{kz} \\
&\qquad \times \bar{U}(z, r)\, dz \\
C(k, r) &= \frac{r}{k}\bar{\eta}_c + \tfrac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{-\infty} k^{-1} e^{kz}\bar{U}(z, r)\, dz.
\end{aligned}\right\} \qquad \ldots(2.18)$$

From (2.13), (2.16) – (2.18) we get,

$$\sqrt{\frac{\pi}{2}}\,\bar{\eta}_c = \frac{-rc}{r^2+m^2}\left[s\sinh kh\int_0^{-\infty} e^{kz}\bar{U}\,dz + \int_0^h \cosh k(h-z)\bar{U}\,dz\right] \qquad \ldots(2.19)$$

$$\begin{aligned}
\sqrt{\frac{\pi}{2}}\,\bar{\phi}_c = k^{-1}\Big[&\int_0^y \sinh k(y-z)\bar{U}\,dz + cs\cosh k(h-y)\int_0^{-\infty} \\
&\times e^{kz}\bar{U}\,dz - c(\sinh ky + s\cosh ky)\int_0^h \cosh k(h-z)\bar{U}\,dz\Big] \\
&- c^2 g(1-s)(r^2+m^2)^{-1}\cosh k(h-y)\Big[s\sinh kh\int_0^{-\infty} e^{kz}\bar{U}\,dz \\
&+ \int_0^h \cosh k(h-z)\bar{U}\,dz\Big]
\end{aligned} \qquad \ldots(2.20)$$

$$\begin{aligned}
\sqrt{\frac{\pi}{2}}\,\bar{\phi}_c = k^{-1}\Big[&\int_0^y \sinh k(y-z)\bar{U}\,dz + c(\cosh ky\cosh kh \\
&- s\sinh ky\sinh kh) \\
&\times \int_0^{-\infty} e^{kz}\bar{U}\,dz - c\,e^{ky}\int_0^n \cosh k(h-z)\bar{U}\,dz\Big] \\
&+ c^2 g(1-s)(r^2+m^2)^{-1}\sinh kh\, e^{ky}\Big[s\sinh kh\int_0^{-\infty}
\end{aligned}$$

(equation continued on p. 1036)

$$\times \; e^{kz} \, \bar{U} \, dz + \int_0^h \cosh k(h-z) \; \bar{U} \, dz] \qquad \ldots(2.21)$$

where $1/c = \cosh kh + s \sinh kh$ and

$$m^2(k) = \frac{(1-s)\,kg}{s + (\coth)\,kh}.$$

The inverse Laplace and cosine Fourier transforms together with the convolution theorem for Laplace transform give :

$$\eta(x, t) = - \; 2/\pi \int_0^\infty \cos kx \, dk \, [\{cs \sinh kh \int_0^{-\infty} e^{kz} \, dz$$

$$+ \; c \int_0^h \cosh k(h-z) \, dz\}. \int_0^t U(z, \tau) \cos m(t-\tau) \, d\tau] \qquad \ldots(2.22)$$

$$\phi(x, y, t) = 2/\pi \int_0^\infty k^{-1} \cos kx \, dk \, [\int_0^y \sinh k(y-z) \, U(z, t) \, dz$$

$$+ \; cs \cosh k(h-y) \int_0^{-\infty} e^{kz} \, U(z, t) \, dz]$$

$$- c\,(\sinh ky + s \cosh ky) \int_0^h \cosh k(h-z) \, U(z, t) \, dz]$$

$$- \; 2/\pi \, g(1-s) \int_0^\infty \cos kx \, dk \, [\{c^2 \cosh k(h-y)$$

$$[s \sinh kh \int_0^{-\infty} e^{kz} \, dz$$

$$+ \int_0^h \cosh k(h-z) \, dz]\}. \int_0^t m^{-1} \sin m(t-\tau) \, U(z, \tau) \, d\tau] \qquad \ldots(2.23)$$

$$\phi'(x, y, t) = 2/\pi \int_0^\infty k^{-1} \cos kx \, dk \, [\int_0^y \sinh k(y-z) \, U(z, t) \, dz$$

$$+ \; c \, (\cosh ky \cosh kh - s \sinh ky \sinh kh) \int_0^{-\infty} e^{kz} \, U(z, t) \, dz$$

$$- \; c \, e^{ky} \int_0^h \cosh k(h-z) \, U(z, t) \, dz]$$

$$+ \; 2/\pi \, g(1-s) \int_0^\infty \cos kx \, dk \, [c^2 \sinh kh \, e^{ky} \, [s \sinh kh \int_0^{-\infty}$$

(*equation continued on p.* 1037)

$$\times\ e^{kz}\, dz + \int_0^h \cosh k\,(h-z)\, dz].\ \int_0^t m^{-1} \sin m\,(t-\tau)$$

$$\times\ U(z, \tau)\, d\tau]. \qquad \text{...(2.24)}$$

In the following we shall consider the velocity of the wavemaker given by

$$U(y, t) = u(y)\, e^{iwt}. \qquad \text{...(2.25)}$$

Now using the particular form of $U(y, t)$ as given in (2.25) in integral representations for $\eta$, $\phi$, $\phi'$ are given by

$$\eta(x, t) = -2/\pi\, e^{iwt} \int_0^\infty (a(k) + s\, b(k)) \cos k\, x\, dk$$

$$\times \int_0^t e^{-iw(t-\tau)} \cos m\,(t-\tau)\, d\tau \qquad \text{...(2.26)}$$

$$\phi(x, y, t) = 2/\pi\, e^{iwt} \int_0^\infty [K(k, y) - (\sinh ky + s \cosh ky)\, u(k)$$

$$+ s \cosh k\,(h-y).\ \frac{b(k)}{\sinh kh}]\, k^{-1} \cos k\, x\, dk - \frac{2g}{\pi}$$

$$\times\ (1-s)\, e^{iwt}.\ \int_0^\infty \frac{c}{m}\ (a(k) + s\, b(k)) \cosh k\,(h-y)$$

$$\times\ \cos k\, x dk.\ \ \int_0^t e^{-iw(t-\tau)} \sin m\,(t-\tau)\, d\tau \qquad \text{...(2.27)}$$

$$\phi'(x, y, t) = 2/\pi\, e^{iwt} \int_0^\infty K(k, y) - e^{ky}\, a(k) + (\cosh ky \cosh kh - s \sinh$$

$$\times\ ky \sinh kh)\ \ \frac{b(k)}{\sinh kh}\ ]\, k^{-1} \cos kx\, dk + \frac{2g}{m}\ (1-s)$$

$$\times\ e^{iwt} \int_0^\infty \frac{c}{m} (a(k) + s\, b(k))\ \sinh kh\, e^{ky}\ \cos kx\, dk \int_0^t$$

$$\times\ e^{-iw(t-\tau)} \sin m\,(t-\tau)\, d\tau \qquad \text{...(2.28)}$$

where

$$K(k, y) = \int_0^y \sinh k\,(y-z)\, u(z)\, dz$$

$$a(k) = c\ \int_0^y \cosh k\,(h-z)\, u(z),\ b(k) = c \sinh kh \int_0^{-\infty} e^{kz}\, u(z)\, dz$$

i. e.

$$\eta(x,t) = \frac{2}{\pi}\int_0^\infty \frac{a(k) + s\,b(k)}{m^2 - w^2}\left[iw \cos m\,t - m \sin mt - iw\, e^{iwt}\right]$$

$$\times \cos kx\, dk \qquad \ldots(2.29)$$

$$\phi(x,y,t) = \frac{2}{\pi}\, e^{iwt} \int_0^\infty [K(k,y) - (\sinh ky + s \cosh ky)\, a(k)$$

$$+ s \cosh k(h-y)\, \frac{b(k)}{\sinh kh}]\, k^{-1} \cos k\,x\, dk$$

$$+ \frac{2}{\pi}\, g(1-s) \int_0^\infty \frac{c\,(a(k) + s\,b(k)) \cosh k(h-y)}{m(m^2 - w^2)}$$

$$\times [m \cos mt + iw \sin mt - m\, e^{iwt}] \cos k\,x\, dk \qquad \ldots(2.30)$$

$$\phi'(x,y,t) = \frac{2}{\pi}\, e^{iwt} \int_0^t [K(k,y) - e^{ky}\, a(k) + (\cosh ky \cosh kh$$

$$- s \sinh ky \sinh kh)$$

$$\times \frac{b(k)}{\sinh kh}]\, k^{-1} \cos k\,x\, dk$$

$$- \frac{2}{\pi} g(1-s) \int_0^\infty \frac{c\,(a(k) + s\,b(k)) \sinh kh\, e^{ky}}{m(m^2 - w^2)}\, [m \cos$$

$$\times\, mt + iw \sin mt - m\, e^{iwt}] \cos kx\, dk. \qquad \ldots(2.31)$$

## 3. Asymptotic Analysis of Solution

We are interested in the waves after a large time at a large distance. To investigate the principal feature of the wave motion, it sufficies to work only with the interface elevation $\eta(x,t)$.

Write $\eta = I + J$

where

$$I = -\frac{2}{\pi}\, iw\, e^{iwt} \int_0^\infty \frac{a(k) + s\,b(k)}{m^2 - w^2} \cos kx\, dk \qquad \ldots(3.1)$$

$$J = \frac{2}{\pi}\int_0^\infty \frac{a(k) + s\,b(k)}{m^2 - w^2}\,(iw \cos mt - m \sin mt) \cos k\,x\, dl \qquad \ldots(3.2)$$

The first integral represents the steady state solution while the second represents the transient solution. It is convenient to rewrite (3.1) and (3.2) as follows.

$$I = \frac{i}{2\pi} e^{iwt} \sum_{n=1}^{2} I_n, \; J = \frac{i}{2\pi} \sum_{n=1}^{4} J_n$$

where

$$I_1, I_2 = \mp \int_0^\infty \frac{a(k) + s\, b(k)}{m \mp w} (e^{ikx} + e^{-ikx})\, dk$$

$$J_1, J_2 = \int_0^\infty \frac{a(k) + s\, b(k)}{m - w} e^{i(wt \pm kx)}\, dk$$

$$J_3, J_4 = -\int_0^\infty \frac{a(k) + s\, b(k)}{m + w} e^{i(wt \mp kx)}\, dk.$$

The main contribution to the asymtotic value of the above integrals for large $t$ and $x$ comes from the poles and stationary points of the integrands. It is noted that each of the integrals $I_1$, $J_1$ and $J_2$ contains one pole at $k = k_0$ where $k_0$ is the only real positive root of the equation

$$\sqrt{\frac{(1-s)\, gk}{s + \coth kh}} = w. \qquad \ldots(3.3)$$

In addition the integrals $J_2$ and $J_3$ contains are stationary point, at $k = k_1$ which is the root of the equation

$$\frac{dm}{dk} = \frac{x}{t}. \qquad \ldots(3.4)$$

We note that

$$\frac{d^2m}{dk^2} = \frac{m(k)}{4k^2} (s \sinh 2kh + \cosh 2kh - 1)^{-1} [(4h^2 k^2 - \sinh^2 2kh)$$
$$+ 4kh (\sinh 2kh - 2kh \cosh 2kh) - s^2 (\cosh 2kh - 1)^2$$
$$+ s(-8h^2 k^2 \sinh 2kh + 2\,(2kh - \sinh^2 2kh)(\cosh 2kh$$
$$- 1)] < 0.$$

Therefore $dm/dk$ decreases monotonically from $[gh(1-s)]^{1/2}$ to 0 as $k$ varies from 0 to $\infty$. Hence equation (3.4) has only one real root $k_1$. On the otherhand, the integrals $I_2$ and $J_4$ contains neither poles nor statianary points in the range of integration.

Now the contribution from the pole of the integral $I_1$ can be evaluated using the formula for the asymptotic development of the generalised Fourier transform developed by Lighthill[6], that is, if $f(k)$ has a simple pole at $k = k_0$ in $a < k_0 < b$, then as $|x| \to \infty$,

$$\int_a^b f(k)\, e^{ikx}\, dk \sim i\,\pi \operatorname{sgn} x\, e^{ik_0x} \text{ (residue of } f(k) \text{ at } k = k_0)$$

$$+ O\left(\frac{1}{|x|}\right). \qquad \text{...(3.5)}$$

Using this formula, it is easy to see that as $x \to \infty$

$$I \sim \frac{a(k_0) + s\,b(k_0)}{2m'(k_0)}\, e^{iwt}\,(e^{ik_0x} - e^{-ik_0x}) \qquad \text{...(3.6)}$$

where $m'(k_0)$ is the derivative of $m(k)$ at $k = k_0$.

The method of stationary phase[7] can be used to evaluate the transient component of $J$ (that is the contribution from the stationary points)

$$J_{tr} \sim \frac{i}{2\pi}\,(a(k_0) + s\,b(k_0))\sqrt{\frac{2}{t\,|m''(k_1)|}}\left\{\frac{e^{i[tm(k_1)-k_1\alpha-\pi/4]}}{m(k_1)-w}\right.$$

$$\left. - \frac{e^{-i[tm(k_1)-k_1x-\pi/4]}}{m(k_1)+w}\right\} + O\left(\frac{1}{t}\right) \qquad \text{...(3.7)}$$

where $J_{tr}$ denotes the transient part of $J$ for large $t$.

Finally we calculate the contribution to $J$ from its polar singularity. This can easily be estimated by the formula (3.5),

$$J_{\text{polar}} \sim -\frac{a(k_0) + s\,b(k_0)}{2\,m'(k_0)}\, e^{iwt}\,(e^{ik_0x} + e^{-ik_0x}). \qquad \text{...(3.8)}$$

Write

$$\eta = \eta_{st} + \eta_{tr}$$

where $\eta_{st}$ is the steady state solution and $\eta_{tr}$ is the transient component. The first term in $\eta$ is the polar contribution to $I$ and $J$ which is given by

$$\eta_{st}(x, t) = -\frac{a(k_0) - s\,b(k_0)}{m'(k_0)}\, e^{i(wt-k_0x)} + O\left(\frac{1}{x}\right) \qquad \text{...(3.9)}$$

and the transient solution $\eta_{tr}$ is given by (3.7).

## 4. Asymptotic Solution when both the Fluids are of Infinite Depth

If the lower fluid is of infinite depth that is when $h \to \infty$, the functions $a(k)$, $b(k)$, $m(k)$, the pole $k_0$ and the stationary point $k_1$ are all simpler in form and they are given by

$$a(k) = \frac{1}{1+s}\int_0^{\infty} e^{-kz}\, u(z)\, dz,\; b(k) = \frac{1}{1+s}\int_0^{-\infty} e^{kz}\, u(z)\, dz$$

$$m(k) = \sqrt{\frac{1-s}{1+s}\, gk},\; k_0 = \frac{1+s}{1-s}\,\frac{w^2}{g},\; k_1 = \frac{1-s}{1+s}\,\frac{gt^2}{4x^2}\,.$$

Therefore in this case, the asymptotic solution for $\eta(x, t)$ can be obtained independently, or from (3.7) and (3.9) by letting formally $h \to \infty$

$$\eta_{st} \sim -\frac{2w}{g}\,\frac{1+s}{1-s}\,(a(k_0) + s\, b(k_0))\, e^{i(wt-k_0x)} \qquad \ldots(4.1)$$

$$\eta_{tr} = \frac{i}{2\sqrt{\pi}}\sqrt{\frac{1-s}{1+s}\, g}\;\frac{t}{x^{3/2}}\,(a(k_1) + s\, b(k_1))$$

$$\times\left[\frac{e^{\,i\left[\frac{1-s}{1+s}\,\frac{gt^2}{4x} - \frac{\pi}{4}\right]}}{\frac{1-s}{1+s}\,\frac{gt}{2x} - w} - \frac{e^{-i\left[\frac{1-s}{1+s}\,\frac{gt^2}{4x} - \frac{\pi}{4}\right]}}{\frac{1-s}{1+s}\,\frac{gt}{2x} - w}\right]. \qquad \ldots(4.2)$$

We note here that solutions (3.7) and (4.2) is not valid when $k_0 = k_1$ which corresponds for the case of both fluids are of infinite depth at the point $x = \frac{1-s}{1+s}\cdot\frac{gt}{2w}$ where the corresponding solution breaks down. A special device developed by Wurtele[8] is needed to obtain a solution valid at the critical points. In view of the fact that the asymptotic solution for large $t$ is of speciel interest, we shall not work out the solution at $k_0 = k_1$.

So far the entire analysis has been carried out for an arbitrary function $u(y)$ involved in the velocity of the wavemaker given by (2.25). It is of physical interest to take $u(y)$ as

$$u(y) = V e^{-k_0|y|}$$

where $V$ is a real constant. Therefore we have for the case when both fiuids are of infinite depth

$$a(k) = -\, b(k) = V(1 + s^{-1}(k + k_0)^{-l_0}.$$

In this case, the asymptotic solution for the interface elevation $\eta(x, t)$ has the form

$$\eta \sim -\frac{qV}{w}\, e^{i(wt-(w2/yg)x)}$$

$$+\frac{i}{\sqrt{\pi}}\,\frac{q^{5/2}\, g^{3/2}\, Vt\, x^{-3/2}}{w^3\,[(qgt/2Wx)^4 - 1]}\,[\cos(qgt^2/4x - \pi/4)$$

(equation continued on p. 1042)

$$+ i\,(q\,gt/2wx)\ \sin\,(qgt^2/4x - \pi/4)] \qquad \ldots(4.3)$$

where $q = (1 + s)/(1 - s)$.

## 5. Conclusion

The above analysis reveals the faet that the transient solution $\eta_{tr}$ as given by (3.7) and (4.2) decays rapidly to zero as time $t \to \infty$. Thus the ultimate steady-state is established in the limit and is given by (3.9), [4.1). These solutions represent outgoing progressive waves that propagates with phase velocity $\dfrac{w}{k_0}$ and $\dfrac{(1+s)\,g}{(1-s)\,w}$ respectively.

Known results[5] in the absence of the upper fluid can be made evident by putting $s = 0$. It is clear from (4.3) that the phase velocity, the wavelength and the amplitude of the steady state component of $\eta$ are greater than of the corresponding values in the absence of the upper fluid.

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